

Five-Loop Anomalous Dimension of Twist-Two Operators

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Abstract

In this article we calculate the five-loop anomalous dimension of twist-two operators in the planar $\mathcal{N} = 4$ SYM theory. Firstly, using reciprocity, we derive the contribution of the asymptotic Bethe ansatz. Subsequently, we employ the first finite-size correction for the $AdS_5 \times S^5$ sigma model to determine the wrapping correction. The anomalous dimension found in this way passes all known tests provided by the NLO BFKL equation and double-logarithmic constraints. This result thus furnishes an *infinite* number of experimental data for testing the veracity of the recently proposed spectral equations for planar AdS/CFT correspondence.

1 Introduction and Summary

Integrable structures appeared for the first time in four-dimensional gauge field theories in the context of high-energy scattering in quantum chromodynamics (QCD). In the Regge limit the scattering amplitudes of colorless particles are dominated by the exchange of two effective particles, termed reggeized gluons. A compound of two of these particles is frequently called the *pomeron* and corresponds to the leading asymptotics of the scattering amplitudes in the Regge limit. In the infinite energy limit $s \rightarrow \infty$, however, the Froissart bound on the cross-section applies and further corrections must be taken into account in order to comply with unitarity. Physically this corresponds to including part of the interactions between *many* reggeized gluons. In the planar limit the Hamiltonian governing the dynamics of these gluons turns out to be integrable [1]. Moreover, it can be decomposed into a holomorphic and an antiholomorphic part, each of which can be identified with the Heisenberg magnet of spin 0, see [2, 3]. The length of this spin chain equals the number of reggeized gluons considered. The pomeron corresponds to the ground state of the length-two spin chain. Although formally the S-matrix of the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory cannot be defined, the analysis of the scattering amplitudes reveals that gluons also reggeize in this case¹. On top of that, the large amount of symmetries significantly simplifies the reggeon Hamiltonian and the Balitsky-Fadin-Kuraev-Lipatov (BFKL) [4, 5, 6] equation, i.e. the eigenvalue equation for the reggeon Hamiltonian, takes a particularly simple form [7].

The planar $\mathcal{N} = 4$ gauge theory exhibits a very remarkable property, absent in most gauge theories². Namely, the dilatation operator is believed to be asymptotically integrable. In the groundbreaking paper [10] the one-loop integrability of the dilatation operator in certain subsectors of the gauge side of the correspondence was discovered. Later on [11], the complete one-loop dilatation operator and the one-loop Bethe equations were written down in [12]. After many non-trivial steps [13] the form of the all-loop asymptotic Bethe equations (ABE) was conjectured [14] up to the so called dressing factor, which only contributes starting from the four-loop order. Subsequently, relying on the crossing equation proposed in [15] and assuming certain transcendentality properties, it was possible to uniquely fix this factor [16]. This completed the solution to the spectral problem of the planar $\mathcal{N} = 4$ SYM theory in the asymptotic region. The asymptoticity of these equations means that for a generic operator with L constituent fields the corresponding anomalous dimension can be calculated correctly up to the $\mathcal{O}(g^{2L})$ order.

Beyond this order the asymptotic Bethe equations are *not* valid, as it was shown in [17]. In [17] the anomalous dimension of twist-two operators has been calculated in a closed form at fourth order in perturbation theory. The analytic continuation to negative integer values of the spin allowed for a comparison of the results with the predictions of the BFKL equation. As a result, a maximal violation of the BFKL prediction has been found, which unambiguously corroborates the breakdown of the asymptotic integrability. The latter

¹This feature is common to many gauge theories with non-abelian gauge group.

²See, however, [8] and [9] for the one- and two-loop integrability in $\mathfrak{sl}(2)$ sector of QCD.

was established by disregarding the contribution of a certain class of Feynman diagrams while studying the structure of mixing of the operators. These diagrams are commonly referred to as wrapping diagrams, the reason being their topological properties [18]. In the spin chain picture these diagrams correspond to the interactions wrapping around the spin chain. Since the interaction between two nearest neighbours provides a factor of g^2 , the first wrapping diagram may appear at the order $\mathcal{O}(g^{2L})$, where L is the length of the spin chain. For twist-two (length-two) operators the supersymmetry *delays* the wrapping interactions to the four-loop order. In that sense the breakdown of the ABE was expected from this point of view.

In the seminal paper [19] it was proposed to evaluate the Lüscher corrections, previously proposed for the $AdS_5 \times S^5$ sigma model in [20], to calculate the leading wrapping corrections to the simplest unprotected operator in the $\mathfrak{sl}(2)$ sector, the Konishi operator. The result remarkably coincides with very complicated field theory computations [21, 22]. In the same way the four-loop wrapping correction for the whole family of twist-two operators has been determined [23]. When added to the ABA result, it restores the agreement with the BFKL prediction! Recently, the same correction has been employed to calculate the five-loop anomalous dimension of the Konishi operator [24].

As for other integrable sigma models, the full solution to the spectral problem is believed to be given in terms of a set of TBA equations. There has been recently a lot of progress in this direction [25]-[29] and ultimately the Y-system [30] and the TBA equations for the ground state [31]-[33] have been formulated. The authors of [32] have also proposed the TBA equations for excited states in the $\mathfrak{sl}(2)$ sector.

It is thus necessary to check these all-loop proposals at higher orders of perturbation theory³. In this article we derive the *full* five-loop anomalous dimension of twist-two operators, which may serve as the sought-for “experimental data” to verify the proposed spectral equations of $\mathcal{N} = 4$ SYM. It can be split into the ABA part and the contribution of the wrapping interactions

$$\gamma_{10}(M) = \gamma_{10}^{\text{ABA}}(M) + \Delta_w(M). \quad (1.1)$$

We present the explicit expression for $\gamma_{10}^{\text{ABA}}(M)$ in appendix A. For the wrapping correction

³In [34] TBA equations for the Konishi operator have been studied numerically at many different values of the coupling constant. In particular, the strong-coupling expansion of the scaling dimension has been conjectured, which structurally but not quantitatively coincides with the string theory computation [35]. As shown in [36], the contribution of the non-wrapping diagrams violates the structure of the strong-coupling expansion and thus analytic derivation of the strong-coupling expansion seems necessary. This is even more urging in view of the fact that the TBA equations seem to predict infinitely many singular values of the coupling constant, see [37].

$\Delta_w(M)$ we have found

$$\begin{aligned}
\Delta_w = & 13440 \zeta(7) S_1^2 - 1536 \zeta(3)^2 S_1^3 + 2560 \zeta(5) S_1 \left(3 S_1 (2 S_{-2} + S_2) - S_1^3 + S_{-3} + S_3 - 2 S_{-2,1} \right) \\
& + 1024 \zeta(3) S_1 \left(-2 S_1^3 S_{-2} + 2 S_1^2 (2 S_{-3} + 3 S_3) + S_1 (4 S_{-2}^2 + 6 S_2 S_{-2} + 3 S_{-4} - S_4 \right. \\
& \left. - 2 (S_{-3,1} - 2 S_{-2,-2} + S_{-2,2} + S_{3,1} - 2 S_{-2,1,1})) + 2 S_{-2} (S_{-3} + S_3 - 2 S_{-2,1}) \right) \\
& - 1024 S_1 \left((S_1 (3 S_2 + 2 S_{-2}) + S_{-3} + S_3 - 2 S_{-2,1} - S_1^3) (S_{-5} - S_5 + 2 S_{-2,-3} - 2 S_{3,-2} \right. \\
& + 2 S_{4,1} - 4 S_{-2,-2,1}) + 2 S_1^2 (2 S_{-6} - 2 S_6 - S_{-4,-2} + 2 S_{-3,-3} + 3 S_{-2,-4} + S_{-2,4} \\
& - 2 S_{3,-3} - 2 S_{4,-2} + S_{4,2} + 4 S_{5,1} - 4 S_{-3,-2,1} - 4 S_{-2,-3,1} - 2 S_{-2,-2,-2} - 2 S_{-2,-2,2}) \\
& + S_1 (5 S_{-7} - 5 S_7 - 4 S_{-6,1} + 4 S_{-5,-2} - S_{-5,2} + 3 S_{-4,-3} + S_{-3,-4} - S_{-3,4} + 8 S_{-2,-5} \\
& - 6 S_{-2,5} - 4 S_{3,-4} + 2 S_{3,4} - 8 S_{4,-3} + 3 S_{4,3} - 6 S_{5,-2} + S_{5,2} + 6 S_{6,1} + 2 S_{-5,1,1} \\
& - 6 S_{-4,-2,1} - 2 S_{-3,-3,1} + 2 S_{-3,-2,-2} - 2 S_{-3,1,-3} - 8 S_{-2,-4,1} + 6 S_{-2,-3,-2} - 2 S_{-2,-3,2} \\
& + 14 S_{-2,-2,-3} - 6 S_{-2,-2,3} - 2 S_{-2,1,-4} + 2 S_{-2,1,4} - 2 S_{-2,2,-3} - 4 S_{-2,3,-2} + 10 S_{-2,4,1} \\
& + 2 S_{3,-3,1} - 4 S_{3,-2,-2} + 2 S_{3,-2,2} + 2 S_{3,1,-3} + 2 S_{3,2,-2} + 10 S_{4,-2,1} + 6 S_{4,1,-2} - 2 S_{4,1,2} \\
& - 2 S_{4,2,1} - 2 S_{5,1,1} + 4 S_{-3,1,-2,1} + 4 S_{-2,-3,1,1} - 20 S_{-2,-2,-2,1} - 8 S_{-2,-2,1,-2} \\
& + 4 S_{-2,-2,1,2} + 4 S_{-2,-2,2,1} + 4 S_{-2,1,-3,1} - 4 S_{-2,1,-2,-2} + 4 S_{-2,1,1,-3} + 4 S_{-2,2,-2,1} \\
& \left. - 4 S_{3,-2,1,1} - 4 S_{3,1,1,-2} + 4 S_{4,1,1,1} - 8 S_{-2,-2,1,1,1} - 8 S_{-2,1,1,-2,1}) \right). \tag{1.2}
\end{aligned}$$

The analytic continuation of the full anomalous dimension (1.1) to the BFKL pole $M = -1 + \omega$ is tedious but feasible. One finds *full agreement* with the LO and NLO BFKL prediction! Also the double-logarithmic constraints are satisfied. This is a very non-trivial check of our findings and a convincing evidence that the first Lüscher correction may be applied at the five-loop order without any modifications. In particular, the successful comparison with the BFKL equation indirectly supports the correctness of the $M = 2$ result, i.e. the five-loop anomalous dimension of the Konishi operator, found in [24].

This paper is structured as follows. In section 2 we review the basic facts about the structure of the anomalous dimension of twist-two operators. In section 3 we discuss the known constraints on the five-loop anomalous dimension. They provide a possibility of an *independent* verification of our findings. We derive the contribution of the asymptotic Bethe ansatz to the five-loop anomalous dimension in section 4. At the end of the section, we also perform analytic continuations of the result relevant from the perspective of the constraints discussed in section 3. Finally, in section 5 we employ the first Lüscher correction to calculate the contribution of the wrapping diagrams and subsequently check it against the constraints discussed in section 3. Some of the lengthy results and formulas are delegated to appendices. Aware of the inconvenience that typing in of our results may cause, we have decided to set up a webpage with *Mathematica* notebooks containing main results obtained in this article. It can be found under <http://thd.pnpi.spb.ru/~velizh/5loop/>.

2 The structure of the anomalous dimension

Twist-two operators belong to the $\mathfrak{sl}(2)$ sub-sector of the full theory. The highest-weight representative consists of two scalar fields \mathcal{Z} and M covariant derivatives \mathcal{D}

$$\text{Tr} \left(\mathcal{Z} \mathcal{D}^M \mathcal{Z} \right) + \dots \quad (2.1)$$

Note that proper eigenstates of the dilatation operator are linear combinations of different distributions of the derivatives over the scalar fields. It is well-known that the anomalous dimension of these operators governs the leading breaking of the Bjorken scaling. In the spin chain picture they are identified with the states of the non-compact $\mathfrak{sl}(2)$ spin $= -\frac{1}{2}$ length-two Heisenberg magnet with M excitations. The cyclicity of the trace eliminates states with odd values of M . For each even M , on the other hand, there is precisely one non-BPS state whose total scaling dimension is

$$\Delta = 2 + M + \gamma(g), \quad \text{with} \quad \gamma(g) = \sum_{\ell=1}^{\infty} \gamma_{2\ell} g^{2\ell}. \quad (2.2)$$

Here, $\gamma(g)$ is the anomalous part of the dimension depending on the coupling constant

$$g^2 = \frac{\lambda}{16\pi^2}, \quad (2.3)$$

and $\lambda = N g_{\text{YM}}^2$ is the 't Hooft coupling constant. The anomalous dimension $\gamma(g)$ may be determined to the three-loop order $\mathcal{O}(g^6)$ with help of the asymptotic Bethe ansatz [38]. We will briefly discuss the $\mathfrak{sl}(2)$ Bethe equations in section 4.

Based on the observations made in [7, 39], the authors of [17] have formulated the principle of maximal transcendentality. It assumes that at each order of the perturbation theory ℓ the anomalous dimension of twist-two operators is expressed through the generalised harmonic sums of the order $(2\ell - 1)$, or through the products of zeta functions and harmonic sums for which the sum of the arguments of the zeta functions and the orders of the harmonic sums is equal to $(2\ell - 1)$. The generalised harmonic sums are defined by the following recursive procedure (see [40])

$$S_a(M) = \sum_{j=1}^M \frac{(\text{sgn}(a))^j}{j^{|a|}}, \quad S_{a_1, \dots, a_n}(M) = \sum_{j=1}^M \frac{(\text{sgn}(a_1))^j}{j^{|a_1|}} S_{a_2, \dots, a_n}(j). \quad (2.4)$$

The order ℓ of each sum S_{a_1, \dots, a_n} is given by the sum of the absolute values of its indices

$$\ell = |a_1| + \dots + |a_n|, \quad (2.5)$$

and the order of a product of harmonic sums is equal to the sum of the orders of its constituents. The canonical basis of the harmonic sums of ℓ -th order is spanned by

$$\{S_{a_{11}}, S_{a_{21}, a_{22}}, \dots, S_{a_{\ell 1}, a_{\ell 2}, \dots, a_{\ell \ell}} : a_{ij} \in \mathbb{Z}, \\ \ell = |a_{11}| = |a_{21}| + |a_{22}| = \dots = |a_{\ell 1}| + |a_{\ell 2}| + \dots + |a_{\ell \ell}|\}, \quad (2.6)$$

where the M dependence of the sums is implicit. Each ℓ -th order product of harmonic sums can be decomposed in this basis. The one-, two- and three-loop anomalous dimensions are simple combinations of the sums (2.4) and may be found in [41, 39]. One can check [38] and even prove [42] that they coincide with the corresponding results of the ABE. This furnishes evidence of the validity of asymptotic integrability in the $\mathfrak{sl}(2)$ sector up to the $\mathcal{O}(g^6)$ order in perturbation theory. In [17] a four-loop result, as predicted by the ABE, has been found. Also at this order the principle of maximal transcendentality applies, even when the correct wrapping contribution of [23] is added. We thus assume it to hold at the fifth order of perturbation theory as well.

A curious symmetry of the anomalous dimension of twist-two operators is the so-called reciprocity [43, 44, 45], which is a generalisation of the Gribov-Lipatov one-loop reciprocity. In the reciprocity-respecting basis of harmonic sums the \mathcal{P} function, which according to the reciprocity relation (4.6) is closely related to the anomalous dimension, takes much simpler form, as was shown for the first three orders of perturbation theory in [44]. Recently, also the four-loop correction has been simplified in this manner [46]. Reciprocity is a hidden symmetry of the ground states of twist-three operators as well, at least up to five-loop order [47, 48]. In this article we propose to determine the five-loop contribution to $\mathcal{P}(g)$ instead of $\gamma(g)$. Moreover, we introduce an equivalent, but simpler, definition of the reciprocity-respecting basis at any loop order. Eventually, the resulting set of harmonic sums that may contribute at any given order ℓ is only a small subgroup of (2.6). Assuming reciprocity to be a symmetry at the fifth order of perturbation theory, we will compute in section 4 the ABA part of the five-loop \mathcal{P} function along the methods proposed in [17]. It is tedious but straightforward task to extract from it the corresponding five-loop anomalous dimension. We present the result in Appendix A.

3 Weak-coupling constraints

In this section we will discuss the known weak-coupling constraints on the five-loop anomalous dimension of twist-two operators. One class of constraints provides the NLO BFKL equation. A complementary set of constraints follow from the double-logarithmic behaviour of the amplitudes.

The relation between the anomalous dimension of twist-two operators and the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [4, 5, 6] and its next-to-leading order (NLO) generalisation [49, 50] emerges upon analytic continuation of the function $\gamma(g, M)$, and therefore at weak coupling of each of the $\gamma_{2\ell}(M)$, to complex values of M . This is straightforward in the one-loop case since

$$\gamma_2(M) = 8g^2 S_1(M) = 8g^2 (\Psi(M+1) - \Psi(1)) , \quad (3.1)$$

where $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the digamma function. The foundations for analytic continuation of more complicated harmonic sums have been laid in [51]. At any loop order one expects singularities at all *negative integer* values of M . The first in this series of singular

points,

$$M = -1 + \omega, \quad (3.2)$$

corresponds to the aforementioned pomeron. In the above formula ω should be considered infinitesimally small. The BFKL equation relates $\gamma(g)$ and g in the vicinity of the point $M = -1 + \omega$. It predicts that, if expanded in g , the ℓ -loop anomalous dimension $\gamma_{2\ell}(\omega)$ exhibits poles in ω . Moreover, the residues and the order of the poles can be derived directly from the BFKL equation. The BFKL equation has been formulated up to the next-to-leading order (NLO) in the logarithmic expansion and determines the leading and next-to-leading poles of $\gamma_{2\ell}(\omega)$. Please refer to [17] for details and further explanations. The NLO-BFKL equation for twist-two operators in the dimensional reduction scheme can be written as follows

$$\frac{\omega}{-4g^2} = \chi(\gamma) - g^2 \delta(\gamma), \quad (3.3)$$

where

$$\chi(\gamma) = \Psi\left(-\frac{\gamma}{2}\right) + \Psi\left(1 + \frac{\gamma}{2}\right) - 2\Psi(1), \quad (3.4)$$

$$\begin{aligned} \delta(\gamma) = & 4\chi''(\gamma) + 6\zeta(3) + 2\zeta(2)\chi(\gamma) + 4\chi(\gamma)\chi'(\gamma) \\ & - \frac{\pi^3}{\sin \frac{\pi\gamma}{2}} - 4\Phi\left(-\frac{\gamma}{2}\right) - 4\Phi\left(1 + \frac{\gamma}{2}\right). \end{aligned} \quad (3.5)$$

The function $\Phi(\gamma)$ is given by

$$\Phi(\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\gamma)^2} \left[\Psi(k+\gamma+1) - \Psi(1) \right]. \quad (3.6)$$

Upon using the expansion (2.2), one easily determines the leading singularity structure

$$\begin{aligned} \gamma = & (2 + 0\omega + \mathcal{O}(\omega^2)) \left(\frac{-4g^2}{\omega} \right) - (0 + 0\omega + \mathcal{O}(\omega^2)) \left(\frac{-4g^2}{\omega} \right)^2 \\ & + (0 + \zeta(3)\omega + \mathcal{O}(\omega^2)) \left(\frac{-4g^2}{\omega} \right)^3 - \left(4\zeta(3) + \frac{5}{4}\zeta(4)\omega + \mathcal{O}(\omega^2) \right) \left(\frac{-4g^2}{\omega} \right)^4 \\ & - \left(0 + \left(2\zeta(2)\zeta(3) + 16\zeta(5) \right) \omega + \mathcal{O}(\omega^2) \right) \left(\frac{-4g^2}{\omega} \right)^5 \pm \dots \end{aligned} \quad (3.7)$$

Another class of constraints on the five-loop anomalous dimension provide the constraints following from the double-logarithmic asymptotics of the scattering amplitudes. In QED and QCD this phenomenon was extensively studied in [52, 53] and [54, 55, 56] (see also [57, 58]). It amounts to summing up the leading terms $\sim (\alpha \ln^2 s)^n$ in all orders of perturbation theory. Please refer to [7, 50] for the discussion of this limit in the case of the

$\mathcal{N} = 4$ gauge theory. The resulting constraints allow to determine the leading singularity at $M = -2 + \omega$

$$\begin{aligned}\gamma &= -\omega + \omega \sqrt{1 - \frac{16g^2}{\omega^2}} \\ &= 2 \frac{(-4g^2)}{\omega} - 2 \frac{(-4g^2)^2}{\omega^3} + 4 \frac{(-4g^2)^3}{\omega^5} - 10 \frac{(-4g^2)^4}{\omega^7} + 28 \frac{(-4g^2)^5}{\omega^9} - \dots\end{aligned}\quad (3.8)$$

4 The Five-Loop Anomalous Dimension from Bethe Ansatz

The long-range asymptotic Bethe equations for the $\mathfrak{sl}(2)$ operators can be found directly from the full set of the asymptotic Bethe equations proposed in [14, 16]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp(2i\theta(u_k, u_j)), \quad \prod_{k=1}^M \frac{x_k^+}{x_k^-} = 1. \quad (4.1)$$

There are M equations for $k = 1, \dots, M$ which need to be solved for the Bethe roots u_k . The variables x_k^\pm are related to u_k through Zhukovsky map

$$x_k^\pm = x(u_k^\pm), \quad u^\pm = u \pm \frac{i}{2}, \quad x(u) = \frac{u}{2} \left(1 + \sqrt{1 - 4 \frac{g^2}{u^2}}\right). \quad (4.2)$$

The function $\theta(u, v)$ is the celebrated dressing phase and has been conjectured in [16]. To the fifth order in perturbation theory it is sufficient to write

$$\theta(u_k, u_j) = (4\zeta(3)g^6 - 40\zeta(5)g^8) (q_2(u_k)q_3(u_j) - q_3(u_k)q_2(u_j)) + \mathcal{O}(g^{10}), \quad (4.3)$$

where $q_r(u)$ are the eigenvalues of the conserved magnon charges, see [14]. Note that (4.1) has in general many solutions corresponding to different eigenstates of the dilatation operator. Once a solution has been found, the corresponding asymptotic all-loop anomalous dimension is given by

$$\gamma^{\text{ABA}}(g) = 2g^2 \sum_{k=1}^M \left(\frac{i}{x_k^+} - \frac{i}{x_k^-} \right). \quad (4.4)$$

It is related to the *full* anomalous dimension through

$$\gamma(g) = \gamma^{\text{ABA}}(g) + \Delta_w(g). \quad (4.5)$$

The second term is the contribution of the aforementioned wrapping diagrams and for a state of length L it should be taken into account starting from the order $\mathcal{O}(g^{2L+4})$. In this section we determine $\gamma^{\text{ABA}}(g)$ at the fifth order in perturbation theory.

Assuming the transcendentality principle discussed in section 2 together with several other observations related to the nested harmonic sums that cannot contribute, *cf.* [17], the basis (2.6) reduces to 1500 sums, from which 108 may be attributed to the contribution of the dressing phase. Thus, following the method discussed in [17], one would have to expand γ_{10}^{ABA} in this basis and determine the coefficients by means of solving the $\mathfrak{sl}(2)$ asymptotic Bethe equations (4.1) to the precision which would allow to *rationalise* $\gamma_{10}^{\text{ABA}}(M)$ for $M = 1, \dots, 1392^4$. Despite the fact that the one-loop solution is known [17] and that the derivation of the higher-order corrections is merely a linear problem, it is beyond the computational threshold to determine γ_{10}^{ABA} for such high values of M . Fortunately, thanks to the reciprocity [43, 44] the basis may be further reduced to 256 sums, which renders the computation feasible! The asymptotic anomalous dimension, however, is not the right quantity to look at if one wants to make use of this simplification. Indeed, let us define a function $\mathcal{P}^{\text{ABA}}(N)$ [45], such that

$$\gamma^{\text{ABA}}(M) = \mathcal{P}^{\text{ABA}} \left(M + \frac{1}{2} \gamma^{\text{ABA}}(M) \right). \quad (4.6)$$

If not supplemented by further constraints on the structure of $\mathcal{P}^{\text{ABA}}(N)$, this relation is trivial. The reciprocity constrains $\mathcal{P}^{\text{ABA}}(N)$ to

$$\mathcal{P}^{\text{ABA}}(M) = \sum_{\ell \geq 0} \frac{a_\ell(\log J^2)}{J^{2\ell}}, \quad J^2 = M(M+1) \gg 1, \quad (4.7)$$

with the functions $a_\ell(N)$ being polynomials. Upon substituting the perturbative expansion (2.2), one finds to the five-loop order

$$\mathcal{P}^{\text{ABA}}(M) = g^2 \mathcal{P}_2(M) + g^4 \mathcal{P}_4(M) + g^6 \mathcal{P}_6(M) + g^8 \mathcal{P}_8(M) + g^{10} \mathcal{P}_{10}(M) + \dots, \quad (4.8)$$

⁴Although the states for odd values of M do not exist, one can still find the corresponding solutions of the Bethe equations. They, however, do not satisfy the momentum constraint. It was argued in [17] that $\gamma_{2l}^{\text{ABA}}(M)$ may be consistently continued with respect to the ABE to odd values of M .

with the coefficients $\mathcal{P}_{2i}(M)$, $i = 1, \dots, 5$ taking the following form

$$\mathcal{P}_2(M) = \gamma_2, \quad (4.9)$$

$$\mathcal{P}_4(M) = \gamma_4 - \frac{1}{2}\dot{\gamma}_2\gamma_2, \quad (4.10)$$

$$\mathcal{P}_6(M) = \gamma_6 + \frac{1}{4}\gamma_2\dot{\gamma}_2^2 - \frac{1}{2}\gamma_4\dot{\gamma}_2 + \frac{1}{8}\ddot{\gamma}_2\gamma_2^2 - \frac{1}{2}\dot{\gamma}_4\gamma_2, \quad (4.11)$$

$$\begin{aligned} \mathcal{P}_8^{\text{rational}}(M) = & \gamma_8^{\text{rational}} - \frac{1}{8}\gamma_2\dot{\gamma}_2^3 + \frac{1}{4}\gamma_4\dot{\gamma}_2^2 - \frac{3}{16}\ddot{\gamma}_2\gamma_2^2\dot{\gamma}_2 + \frac{1}{2}\dot{\gamma}_4\gamma_2\dot{\gamma}_2 - \frac{1}{2}\gamma_6\dot{\gamma}_2 - \frac{1}{48}\ddot{\gamma}_2\gamma_2^3 \\ & + \frac{1}{8}\ddot{\gamma}_4\gamma_2^2 - \frac{1}{2}\dot{\gamma}_6\gamma_2 - \frac{1}{2}\dot{\gamma}_4\gamma_4 + \frac{1}{4}\ddot{\gamma}_2\gamma_2\gamma_4, \end{aligned} \quad (4.12)$$

$$\begin{aligned} \mathcal{P}_{10}^{\text{rational}}(M) = & \gamma_{10}^{\text{rational}} + \frac{1}{16}\gamma_2\dot{\gamma}_2^4 - \frac{1}{8}\gamma_4\dot{\gamma}_2^3 + \frac{3}{16}\ddot{\gamma}_2\gamma_2^2\dot{\gamma}_2^2 - \frac{3}{8}\dot{\gamma}_4\gamma_2\dot{\gamma}_2^2 + \frac{1}{4}\gamma_6\dot{\gamma}_2^2 \\ & + \frac{1}{24}\ddot{\gamma}_2\gamma_2^3\dot{\gamma}_2 - \frac{3}{16}\ddot{\gamma}_4\gamma_2^2\dot{\gamma}_2 + \frac{1}{2}\dot{\gamma}_6\gamma_2\dot{\gamma}_2 + \frac{1}{2}\dot{\gamma}_4\gamma_4\dot{\gamma}_2 - \frac{3}{8}\ddot{\gamma}_2\gamma_2\gamma_4\dot{\gamma}_2 - \frac{1}{2}\gamma_8^{\text{rational}}\dot{\gamma}_2 \\ & + \frac{1}{384}\ddot{\gamma}_2\gamma_2^4 + \frac{1}{32}\ddot{\gamma}_2^2\gamma_2^3 - \frac{1}{48}\ddot{\gamma}_4\gamma_2^3 + \frac{1}{8}\ddot{\gamma}_6\gamma_2^2 - \frac{3}{16}\ddot{\gamma}_2\dot{\gamma}_4\gamma_2^2 + \frac{1}{8}\ddot{\gamma}_2\gamma_4^2 + \frac{1}{4}\dot{\gamma}_4^2\gamma_2 \\ & - \frac{1}{2}\dot{\gamma}_8^{\text{rational}}\gamma_2 - \frac{1}{16}\ddot{\gamma}_2\gamma_2^2\gamma_4 - \frac{1}{2}\dot{\gamma}_6\gamma_4 + \frac{1}{4}\ddot{\gamma}_4\gamma_2\gamma_4 - \frac{1}{2}\dot{\gamma}_4\gamma_6 + \frac{1}{4}\ddot{\gamma}_2\gamma_2\gamma_6, \end{aligned} \quad (4.13)$$

$$\mathcal{P}_8^{\zeta(3)}(M) = \gamma_8^{\zeta(3)}, \quad (4.14)$$

$$\mathcal{P}_{10}^{\zeta(3)}(M) = \gamma_{10}^{\zeta(3)} - \frac{1}{2}\gamma_8^{\zeta(3)}\dot{\gamma}_2 - \frac{1}{2}\dot{\gamma}_8^{\zeta(3)}\gamma_2, \quad (4.15)$$

$$\mathcal{P}_{10}^{\zeta(5)}(M) = \gamma_{10}^{\zeta(5)}. \quad (4.16)$$

Here, dots over γ indicate derivatives of the harmonic sums with respect to their indices (see [43, 44]), while $\gamma_i^{\text{rational}}$, $\gamma_i^{\zeta(3)}$ and $\gamma_i^{\zeta(5)}$ denote respectively the rational, $\zeta(3)$ and $\zeta(5)$ parts of the anomalous dimension γ_i . Note that $\mathcal{P}_{2i}(M)$ depend linearly on γ_{2i}^{ABA} . The functions \mathcal{P}_8 may be found in [46]. Due to the property (4.7), only certain combinations of harmonic sums may contribute to \mathcal{P}_{2i} . We discuss the basis of harmonic sums respecting (4.7) in D. It turns out that at the five-loop order only 256 of these sums need to be taken into account. With computational effort of more than 400 hours of computer time, we have found the following result for $\mathcal{P}_{10}(M)$

$$\begin{aligned}
\frac{\mathcal{P}_{10}^{\text{rational}}}{128} = & -5 \mathbb{S}_{2,2,5} - \mathbb{S}_{2,6,1} + 19 \mathbb{S}_{3,1,5} - 20 \mathbb{S}_{3,2,4} + 21 \mathbb{S}_{4,1,4} - 24 \mathbb{S}_{4,2,3} + 25 \mathbb{S}_{5,1,3} \\
& -18 \mathbb{S}_{5,2,2} + 7 \mathbb{S}_{6,1,2} - 4 \mathbb{S}_{6,2,1} - 2 \mathbb{S}_{1,1,2,5} + 2 \mathbb{S}_{1,1,6,1} - 2 \mathbb{S}_{1,2,1,5} - \mathbb{S}_{1,2,2,4} \\
& + \mathbb{S}_{1,2,3,3} + \mathbb{S}_{1,2,4,2} - 6 \mathbb{S}_{1,2,5,1} + 23 \mathbb{S}_{1,3,1,4} - 24 \mathbb{S}_{1,3,2,3} - \mathbb{S}_{1,3,4,1} + 23 \mathbb{S}_{1,4,1,3} \\
& -20 \mathbb{S}_{1,4,2,2} - \mathbb{S}_{1,4,3,1} + 13 \mathbb{S}_{1,5,1,2} - 12 \mathbb{S}_{1,5,2,1} + 6 \mathbb{S}_{1,6,1,1} - 2 \mathbb{S}_{2,1,1,5} + 5 \mathbb{S}_{2,1,2,4} \\
& + \mathbb{S}_{2,1,3,3} + \mathbb{S}_{2,1,4,2} - 5 \mathbb{S}_{2,1,5,1} - 16 \mathbb{S}_{2,2,1,4} + 17 \mathbb{S}_{2,2,2,3} - 2 \mathbb{S}_{2,2,3,2} + 14 \mathbb{S}_{2,2,4,1} \\
& -29 \mathbb{S}_{2,3,1,3} + 25 \mathbb{S}_{2,3,2,2} + 4 \mathbb{S}_{2,3,3,1} - 19 \mathbb{S}_{2,4,1,2} + 20 \mathbb{S}_{2,4,2,1} - 12 \mathbb{S}_{2,5,1,1} + 20 \mathbb{S}_{3,1,1,4} \\
& -22 \mathbb{S}_{3,1,2,3} - 8 \mathbb{S}_{3,1,3,2} + 6 \mathbb{S}_{3,1,4,1} - 26 \mathbb{S}_{3,2,1,3} + 36 \mathbb{S}_{3,2,2,2} - 5 \mathbb{S}_{3,2,3,1} - 6 \mathbb{S}_{3,3,1,2} \\
& + 5 \mathbb{S}_{3,3,2,1} - 2 \mathbb{S}_{3,4,1,1} + 22 \mathbb{S}_{4,1,1,3} - 24 \mathbb{S}_{4,1,2,2} + 6 \mathbb{S}_{4,1,3,1} - 18 \mathbb{S}_{4,2,1,2} + 18 \mathbb{S}_{4,2,2,1} \\
& -2 \mathbb{S}_{4,3,1,1} + 14 \mathbb{S}_{5,1,1,2} - 10 \mathbb{S}_{5,1,2,1} - 14 \mathbb{S}_{5,2,1,1} + 8 \mathbb{S}_{6,1,1,1} + 4 \mathbb{S}_{1,1,1,1,5} - 6 \mathbb{S}_{1,1,1,2,4} \\
& -2 \mathbb{S}_{1,1,1,3,3} - 2 \mathbb{S}_{1,1,1,4,2} + 6 \mathbb{S}_{1,1,1,5,1} - 4 \mathbb{S}_{1,1,2,1,4} + 4 \mathbb{S}_{1,1,2,2,3} + 5 \mathbb{S}_{1,1,2,3,2} \\
& -13 \mathbb{S}_{1,1,2,4,1} + 24 \mathbb{S}_{1,1,3,1,3} - 20 \mathbb{S}_{1,1,3,2,2} - 5 \mathbb{S}_{1,1,3,3,1} + 16 \mathbb{S}_{1,1,4,1,2} - 19 \mathbb{S}_{1,1,4,2,1} \\
& + 12 \mathbb{S}_{1,1,5,1,1} - 4 \mathbb{S}_{1,2,1,1,4} + 7 \mathbb{S}_{1,2,1,2,3} + 4 \mathbb{S}_{1,2,1,3,2} - 8 \mathbb{S}_{1,2,1,4,1} - 19 \mathbb{S}_{1,2,2,1,3} \\
& + 9 \mathbb{S}_{1,2,2,2,2} + 24 \mathbb{S}_{1,2,2,3,1} - 22 \mathbb{S}_{1,2,3,1,2} + 31 \mathbb{S}_{1,2,3,2,1} - 22 \mathbb{S}_{1,2,4,1,1} + 22 \mathbb{S}_{1,3,1,1,3} \\
& -24 \mathbb{S}_{1,3,1,2,2} + 6 \mathbb{S}_{1,3,1,3,1} - 20 \mathbb{S}_{1,3,2,1,2} + 23 \mathbb{S}_{1,3,2,2,1} - 6 \mathbb{S}_{1,3,3,1,1} + 16 \mathbb{S}_{1,4,1,1,2} \\
& -11 \mathbb{S}_{1,4,1,2,1} - 22 \mathbb{S}_{1,4,2,1,1} + 16 \mathbb{S}_{1,5,1,1,1} - 4 \mathbb{S}_{2,1,1,1,4} + 7 \mathbb{S}_{2,1,1,2,3} + 4 \mathbb{S}_{2,1,1,3,2} \\
& -8 \mathbb{S}_{2,1,1,4,1} + 3 \mathbb{S}_{2,1,2,1,3} - 11 \mathbb{S}_{2,1,2,2,2} + 14 \mathbb{S}_{2,1,2,3,1} - 9 \mathbb{S}_{2,1,3,1,2} + 16 \mathbb{S}_{2,1,3,2,1} \\
& -12 \mathbb{S}_{2,1,4,1,1} - 16 \mathbb{S}_{2,2,1,1,3} + 13 \mathbb{S}_{2,2,1,2,2} + 5 \mathbb{S}_{2,2,1,3,1} + 28 \mathbb{S}_{2,2,2,1,2} - 67 \mathbb{S}_{2,2,2,2,1} \\
& + 32 \mathbb{S}_{2,2,3,1,1} - 22 \mathbb{S}_{2,3,1,1,2} + 15 \mathbb{S}_{2,3,1,2,1} + 31 \mathbb{S}_{2,3,2,1,1} - 23 \mathbb{S}_{2,4,1,1,1} + 20 \mathbb{S}_{3,1,1,1,3} \\
& -22 \mathbb{S}_{3,1,1,2,2} + 6 \mathbb{S}_{3,1,1,3,1} - 8 \mathbb{S}_{3,1,2,1,2} + 2 \mathbb{S}_{3,1,2,2,1} + 6 \mathbb{S}_{3,1,3,1,1} - 20 \mathbb{S}_{3,2,1,1,2} \\
& + 15 \mathbb{S}_{3,2,1,2,1} + 25 \mathbb{S}_{3,2,2,1,1} - 7 \mathbb{S}_{3,3,1,1,1} + 16 \mathbb{S}_{4,1,1,1,2} - 11 \mathbb{S}_{4,1,1,2,1} - 11 \mathbb{S}_{4,1,2,1,1} \\
& -23 \mathbb{S}_{4,2,1,1,1} + 16 \mathbb{S}_{5,1,1,1,1} - 4 \mathbb{S}_{1,1,1,1,2,3} - 4 \mathbb{S}_{1,1,1,1,3,2} + 8 \mathbb{S}_{1,1,1,1,4,1} + 4 \mathbb{S}_{1,1,1,2,2,2} \\
& -13 \mathbb{S}_{1,1,1,2,3,1} + 18 \mathbb{S}_{1,1,1,3,1,2} - 25 \mathbb{S}_{1,1,1,3,2,1} + 16 \mathbb{S}_{1,1,1,4,1,1} + 3 \mathbb{S}_{1,1,2,1,2,2} \\
& -5 \mathbb{S}_{1,1,2,1,3,1} - 20 \mathbb{S}_{1,1,2,2,1,2} + 53 \mathbb{S}_{1,1,2,2,2,1} - 33 \mathbb{S}_{1,1,2,3,1,1} + 18 \mathbb{S}_{1,1,3,1,1,2} \\
& -11 \mathbb{S}_{1,1,3,1,2,1} - 25 \mathbb{S}_{1,1,3,2,1,1} + 20 \mathbb{S}_{1,1,4,1,1,1} + 3 \mathbb{S}_{1,2,1,1,2,2} - 5 \mathbb{S}_{1,2,1,1,3,1} \\
& + 14 \mathbb{S}_{1,2,1,2,2,1} - 17 \mathbb{S}_{1,2,1,3,1,1} - 18 \mathbb{S}_{1,2,2,1,1,2} + 11 \mathbb{S}_{1,2,2,1,2,1} + 52 \mathbb{S}_{1,2,2,2,1,1} \\
& -27 \mathbb{S}_{1,2,3,1,1,1} + 16 \mathbb{S}_{1,3,1,1,1,2} - 11 \mathbb{S}_{1,3,1,1,2,1} - 11 \mathbb{S}_{1,3,1,2,1,1} - 27 \mathbb{S}_{1,3,2,1,1,1} \\
& + 20 \mathbb{S}_{1,4,1,1,1,1} + 3 \mathbb{S}_{2,1,1,1,2,2} - 5 \mathbb{S}_{2,1,1,1,3,1} + 14 \mathbb{S}_{2,1,1,2,2,1} - 17 \mathbb{S}_{2,1,1,3,1,1} \\
& + 16 \mathbb{S}_{2,1,2,2,1,1} - 11 \mathbb{S}_{2,1,3,1,1,1} - 16 \mathbb{S}_{2,2,1,1,1,2} + 11 \mathbb{S}_{2,2,1,1,2,1} + 11 \mathbb{S}_{2,2,1,2,1,1} \\
& + 45 \mathbb{S}_{2,2,2,1,1,1} - 27 \mathbb{S}_{2,3,1,1,1,1} + 16 \mathbb{S}_{3,1,1,1,1,2} - 11 \mathbb{S}_{3,1,1,1,2,1} - 11 \mathbb{S}_{3,1,1,2,1,1} \\
& -11 \mathbb{S}_{3,1,2,1,1,1} - 27 \mathbb{S}_{3,2,1,1,1,1} + 20 \mathbb{S}_{4,1,1,1,1,1} - 16 \mathbb{S}_{1,1,1,1,2,2,1} + 24 \mathbb{S}_{1,1,1,1,3,1,1} \\
& -28 \mathbb{S}_{1,1,1,2,2,1,1} + 20 \mathbb{S}_{1,1,1,3,1,1,1} - 20 \mathbb{S}_{1,1,2,2,1,1,1} + 20 \mathbb{S}_{1,1,3,1,1,1,1} \\
& -20 \mathbb{S}_{1,2,2,1,1,1,1} + 20 \mathbb{S}_{1,3,1,1,1,1,1} - 20 \mathbb{S}_{2,2,1,1,1,1,1} + 20 \mathbb{S}_{3,1,1,1,1,1,1} . \tag{4.17}
\end{aligned}$$

Table 1: The five-loop function $\mathcal{P}_{10}(M)$.

The functions $\mathcal{P}_{10}^{\zeta(3)}$ and $\mathcal{P}_{10}^{\zeta(5)}$ are easy to determine⁵⁶

$$\begin{aligned} \frac{\mathcal{P}_{10}^{\zeta(3)}}{256} = & 3\mathbb{S}_{1,5} - 4\mathbb{S}_{2,4} - \mathbb{S}_{3,3} - \mathbb{S}_{4,2} + 3\mathbb{S}_{5,1} + 2\mathbb{S}_{1,1,4} - 4\mathbb{S}_{1,2,3} - 2\mathbb{S}_{1,3,2} + 2\mathbb{S}_{1,4,1} \\ & - \mathbb{S}_{2,1,3} + 5\mathbb{S}_{2,2,2} - 2\mathbb{S}_{2,3,1} - 2\mathbb{S}_{3,2,1} + 2\mathbb{S}_{4,1,1} - 3\mathbb{S}_{1,1,2,2} + \mathbb{S}_{1,1,3,1} - \mathbb{S}_{1,2,1,2} \\ & + 2\mathbb{S}_{1,3,1,1} - \mathbb{S}_{2,1,1,2} + \mathbb{S}_{2,1,2,1} + \mathbb{S}_{3,1,1,1}, \end{aligned} \quad (4.18)$$

$$\frac{\mathcal{P}_{10}^{\zeta(5)}}{640} = \mathbb{S}_1(\mathbb{S}_{2,1} - \mathbb{S}_3). \quad (4.19)$$

The resulting asymptotic anomalous dimension, *cf.* table 2 in Appendix A, may now be analytically continued to the BFKL pole $M = -1 + \omega$. With help of the **SUMMER** [40] and **HARMPOL** [61] packages for **FORM** [62] and the **HPL** package [63] for **Mathematica**, we have found

$$\begin{aligned} \gamma^{\text{ABA}} = & \left(\frac{-4g^2}{\omega} \right) \left(2 + 0\omega \right) + \left(\frac{-4g^2}{\omega} \right)^2 \left(0 + 0\omega \right) + \left(\frac{-4g^2}{\omega} \right)^3 \left(0 + \zeta(3)\omega \right) \\ & + \frac{(-4g^2)^4}{\omega^7} \left(-2 + \frac{4\pi^2}{3}\omega^2 - 13\zeta(3)\omega^3 - \frac{8\pi^4}{45}\omega^4 \right) + \frac{(-4g^2)^5}{\omega^9} \times \\ & \left(2 - \frac{7\pi^2}{3}\omega^2 - \frac{39\zeta(3)}{2}\omega^3 + \frac{361\pi^4}{240}\omega^4 + \frac{618\zeta(2)\zeta(3) - 1199\zeta(5)}{8}\omega^5 \right) \\ & \pm \dots, \end{aligned} \quad (4.20)$$

where we have also restated the known results at lower orders. One observes a *maximal violation* of the NLO BFKL prediction (3.4) at the four- and five-loop order; the leading singularity in ω should in both cases be a pole of order four. Instead, we find the leading poles to be of order seven at the four-loop order and of order nine at the next order, thus providing three constraints on the wrapping contribution at the four-loop order and four constraints at five-loops. This counting takes into account the fact that the cancelation of the highest pole in the ABA result is intimately related to the cancelation of the next, lower pole. Please note that after adding the four-loop wrapping correction found in [23] the agreement between (3.7) and (4.20) is restored at the four-loop order!

5 Calculation of the wrapping correction

In this section we calculate the wrapping correction by evaluating the first Lüscher correction at weak-coupling along the lines advocated in [19], [23] and [24]. Note that the second Lüscher correction is not expected to contribute before the order $\mathcal{O}(g^{12})$. Please refer to [24] for further discussion.

⁵⁶The $\zeta(3)$ and $\zeta(5)$ parts of the five-loop ABA contribution to the anomalous dimension have already been found in [59].

⁶Curiously, it seems that at the ℓ -loop order the highest transcendentality contribution of the dressing phase to the anomalous dimension may be found exactly: $-64\beta_{2,3}^\ell S_1(S_3 - S_{-3} + 2S_{-2,1})$.

The total wrapping correction consists of two terms

$$\Delta_w = \Delta_w^F + \Delta_w^{\text{ABA}}. \quad (5.1)$$

The first term, the so-called F-term, reflects the finite-size correction to the dispersion formula, while the second term accounts for the shifts in the magnon rapidities induced by the interactions with the virtual particle.

Following the notation in [24], the F-term integral can be written as

$$\begin{aligned} \Delta_w^F &= - \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left(\frac{z^-}{z^+} \right)^2 S_0(M, q, Q) S_{\sigma}(M, q, Q) S_{\boxplus}(M, q, Q)^2 \\ &\equiv - \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} Y_Q(M, q), \end{aligned} \quad (5.2)$$

where $Y_Q(M, q)$ has the perturbative expansion of the form

$$Y_Q(M, q) = Y_Q^{(8,0)}(M, q) g^8 + \left(Y_Q^{(10,0)}(M, q) + Y_Q^{(8,2)}(M, q) \right) g^{10} + \mathcal{O}(g^{12}). \quad (5.3)$$

The five-loop integrand coming from the F-term has two components. The first one, $Y_Q^{(10,0)}$, follows from expanding the integrand (5.2) to the five-loop order and inserting the one-loop Bethe roots $u_k = u_k^0 + \mathcal{O}(g^2)$. The second term, $Y_Q^{(8,2)}$ accounts for the two-loop corrections to the rapidities of the magnons. Clearly, while calculating $Y_Q^{(8,2)}$, it is sufficient to determine Y_Q^8 as a function of the parameters u_k and subsequently put $u_k = u_k^0 + u_k^2 g^2 + \mathcal{O}(g^4)$

$$Y_Q^8(u_1, \dots, u_M, q) = Y_Q^{(8,0)}(M, q) + g^2 Y_Q^{(8,2)}(M, q) + \mathcal{O}(g^4). \quad (5.4)$$

In order to calculate $Y_Q^{(10,0)}$ we factor out the leading integrand and rewrite it as a sum of the matrix part, the scalar part, the exponential term and the dressing factor

$$\begin{aligned} Y_Q^{(10,0)}(M, q) &= Y_Q^{(8,0)}(M, q) \left[2 \frac{S_{\boxplus}^{(4)}(M, q, Q)}{S_{\boxplus}^{(2)}(M, q, Q)} + \frac{S_0^{(2)}(M, q, Q)}{S_0^{(0)}(M, q, Q)} + \frac{\Upsilon^{(6)}(q, Q)}{\Upsilon^{(4)}(q, Q)} \right. \\ &\quad \left. + S_{\sigma}^{(2)}(M, q, Q) \right]. \end{aligned} \quad (5.5)$$

This section is structured as follows. In sub-sections 5.2 and 5.3 we determine $Y_Q^{(10,0)}$ and $Y_Q^{(8,2)}$. In sub-section 5.4 we will discuss the modification of the quantization condition, leading to Δ_w^{ABA} .

5.1 The four-loop integrand $Y_Q^{(8,0)}$

Before proceeding to the five-loop order we recall the reader the form of $Y_Q^{(8,0)}(M, q)$, as found in [23]

$$Y_Q^{(8,0)}(M, q) = 64 S_1^2 \frac{T_M(q, Q)^2}{R_M(q, Q)} \frac{16}{(q^2 + Q^2)^2}. \quad (5.6)$$

In this formula

$$R_M^{(0)}(q, Q) = P_M^{(0)}\left(\frac{q - i(-1 + Q)}{2}\right) P_M^{(0)}\left(\frac{q + i(-1 + Q)}{2}\right) P_M^{(0)}\left(\frac{q - i(1 + Q)}{2}\right) P_M^{(0)}\left(\frac{q + i(1 + Q)}{2}\right), \quad (5.7)$$

and

$$T_M^{(0)}(q, Q) = \sum_{j=0}^{Q-1} \left(\frac{1}{2j - iq - Q} - (-1)^M \frac{1}{2(j+1) - iq - Q} \right) P_M^{(0)}\left(\frac{q - i(Q-1)}{2} + ij\right). \quad (5.8)$$

The one-loop Baxter function, $P_M^{(0)}$, is a hypergeometric orthogonal polynomial [64]

$$P_M^{(0)}(u) = {}_3F_2 \left(\begin{matrix} -M, & M+1, & \frac{1}{2} + iu \\ & 1, & 1 \end{matrix} \middle| 1 \right). \quad (5.9)$$

5.2 Derivation of $Y^{(10,0)}$

By comparing (5.2) and (5.3), one concludes that the calculation of $Y^{(10,0)}(M, q)$ amounts to expanding $(z^-/z^+)^2$, $S_0(M, q, Q)$, $S_\sigma(M, q, Q)$ and $S_\boxplus(M, q, Q)$ to the next-to-leading order. This is two-loop order for the scalar and the dressing part, whilst the matrix part vanishes for $g = 0$ and needs to be evaluated to three-loop order. On the other hand, the exponential part $(z^-/z^+)^2$ vanishes at the first two orders in g^2 and thus needs to be expanded to the order $\mathcal{O}(g^8)$. Owing to the integration in (5.2), we symmetrise all results with respect to q . The expressions obtained are valid for all *positive integer* values of M .

Scalar Part

Similarly to the case of the Konishi operator [24] the scalar part can be divided into two parts

$$S_0(M, q, Q) = S_0^{(0)}(M, q, Q) + g^2 \left(S_{0rat}^{(2)}(M, q, Q) + S_{0\psi}^{(2)}(M, q, Q) \right) + \mathcal{O}(g^4). \quad (5.10)$$

The first part contains solely rational functions and may be expressed through one-loop Baxter function and its derivative

$$\begin{aligned} \frac{S_{0rat}^{(2)}(M, q, Q)}{S_0^{(0)}(M, q, Q)} &= \frac{4q}{q^2 + Q^2} \left(\frac{P_M^{(0)'}\left(\frac{-q-i(1+Q)}{2}\right)}{P_M^{(0)}\left(\frac{-q-i(1+Q)}{2}\right)} - \frac{P_M^{(0)'}\left(\frac{q-i(1+Q)}{2}\right)}{P_M^{(0)}\left(\frac{q-i(1+Q)}{2}\right)} \right) - 8 S_{-2}(M) \\ &\quad - \frac{32Q}{q^2 + Q^2} S_1(M). \end{aligned} \quad (5.11)$$

The second part consists of polygamma functions and depends on M only through the harmonic number

$$\frac{S_{0\psi}^{(2)}(M, q, Q)}{S_0^{(0)}(q, Q, u)} = 4 S_1(M) \left(\psi\left(\frac{-iq - Q}{2}\right) - \psi\left(\frac{-iq + Q}{2}\right) + \psi\left(\frac{iq - Q}{2}\right) - \psi\left(\frac{iq + Q}{2}\right) \right). \quad (5.12)$$

Dressing Part

In contradistinction to the physical kinematics, the dressing factor governing the scattering of virtual particles on the physical magnons is trivial only at the leading order, i.e. $S_\sigma^{(0)} = 1$, cf. [24]. At the next-to-leading order one finds

$$S_\sigma^{(2)}(M, q, Q) = -4 S_1(M) \left(4\gamma_E + \psi\left(\frac{-iq - Q}{2}\right) + \psi\left(\frac{-iq + Q}{2}\right) + \psi\left(\frac{iq - Q}{2}\right) + \psi\left(\frac{iq + Q}{2}\right) \right). \quad (5.13)$$

When combined with the non-rational part of the scalar factor $S_{0\psi}^{(2)}$, they simplify further to

$$\frac{S_{0\psi}^{(2)}(M, q, Q)}{S_0^{(0)}(q, Q, u)} + S_\sigma^{(2)}(M, q, Q) = -8 S_1(M) \left(2\gamma_E + \psi\left(\frac{-iq + Q}{2}\right) + \psi\left(\frac{iq + Q}{2}\right) \right). \quad (5.14)$$

This is the only part of the integrand containing polygamma functions.

Exponential Part

The exponential part does not depend on M and is straightforward to determine

$$\frac{\Upsilon^{(6)}(q, Q)}{\Upsilon^{(4)}(q, Q)} = -\frac{16}{q^2 + Q^2}. \quad (5.15)$$

The matrix part

The calculation of the matrix part amounts to calculating the supertrace

$$\text{str} \left(\prod_{k=1}^M S_Q^{\text{matrix}}(q, u_k) \right) \quad (5.16)$$

to the three loop-order. This may be systematically achieved by introducing the matrix G that diagonalizes each copy $\hat{S}_{\text{su}(2|2), Q}(q, u_k)$ of the tensor product

$$S_Q^{\text{matrix}}(q, u_k) = \hat{S}_{\text{su}(2|2), Q}(q, u_k) \otimes \hat{S}_{\text{su}(2|2), Q}(q, u_k). \quad (5.17)$$

Explicitly,

$$G \hat{S}_{\text{su}(2|2), Q}(q, u_k) G^{-1} = \hat{S}_{\text{su}(2|2), Q}^{\text{diag}}(q, u_k). \quad (5.18)$$

The key feature of G is that to the order $\mathcal{O}(g^4)$ it *does not* depend on the parameters u_k , i.e. $G = G(q, Q)$. The diagonal matrix $S_Q^{\text{diag}}(q, u_k)$ has the following structure

$$\left\{ \hat{S}_{\text{su}(2|2), Q}^{\text{diag}}(q, u_k) \right\}_{ii} = B_1(i, u_k)\theta(Q-i) + B_2(i-Q, u_k)\theta(2Q-i)\theta(i-Q) - F(i-2Q, u_k)\theta(3Q-i)\theta(i-2Q) - F(i-3Q, u_k)\theta(4Q-i)\theta(i-3Q). \quad (5.19)$$

Here, the index i takes the values $1, \dots, 4Q$. The function θ is the unitstep function. The coefficients B_1, B_2 and F admit the usual perturbative expansion

$$B_1(j, u_k) = B_{1,0}(j, u_k) + g^2 B_{1,2}(j, u_k) + g^4 B_{1,4}(j, u_k) + \mathcal{O}(g^6), \quad (5.20)$$

$$B_2(j, u_k) = B_{2,0}(j, u_k) + g^2 B_{2,2}(j, u_k) + g^4 B_{2,4}(j, u_k) + \mathcal{O}(g^6), \quad (5.21)$$

$$F(j, u_k) = F_0(j, u_k) + g^2 F_2(j, u_k) + g^4 F_4(j, u_k) + \mathcal{O}(g^6). \quad (5.22)$$

In the above formulas, in view of (5.19), the index j takes the values $0, \dots, Q-1$. The explicit form of these coefficients can be found in the Appendix B. The decomposition (5.19) enables a direct evaluation of the supertrace (5.16)

$$\begin{aligned} \text{str} \left(\prod_{k=1}^M \hat{S}_{\text{su}(2|2), Q}^{\text{matrix}}(q, u_k) \right) &= \text{str} \left(\prod_{k=1}^M \hat{S}_{\text{su}(2|2), Q}^{\text{diag}}(q, u_k) \right) = \sum_{j=0}^{Q-1} \left[\prod_{k=1}^M \left(B_{1,0}(j, u_k) \right. \right. \\ &+ g^2 B_{1,2}(j, u_k) + g^4 B_{1,4}(j, u_k) + \mathcal{O}(g^6) \Big) + \prod_{k=1}^M \left(B_{2,0}(j, u_k) + g^2 B_{2,2}(j, u_k) \right. \\ &+ g^4 B_{2,4}(j, u_k) + \mathcal{O}(g^6) \Big) - 2 \prod_{k=1}^M \left(F_0(j, u_k) + g^2 F_2(j, u_k) + g^4 F_4(j, u_k) \right. \\ &\left. \left. + \mathcal{O}(g^6) \right) \right]. \end{aligned} \quad (5.23)$$

Upon expanding this to the order $\mathcal{O}(g^4)$ and simultaneously inserting the one-loop Bethe roots u_k^0 , one finds

$$\left(\text{str} \left(\prod_{k=1}^M \hat{S}_{\text{su}(2|2), Q}^{\text{matrix}}(q, u_k) \right) \right)_{u=u_0} = 0 + g^2 S_{\boxplus}^{(2)}(M, q, Q) + g^4 S_{\boxplus}^{(4)}(M, q, Q) + \mathcal{O}(g^6). \quad (5.24)$$

The absence of the $\mathcal{O}(g^0)$ contribution is non-trivial and is a result of certain functional relations between the coefficients in (5.20)-(5.22). Please note that the higher loop corrections to Bethe roots affect only $S_{\boxplus}^{(2)}(M, q, Q)$ and have already been taken into account in (5.26).

5.3 Calculating $Y_Q^{(8,2)}$

As discussed in the beginning of section 5, the second contribution to the five-loop integrand, $Y_Q^{(8,2)}$, originates from the two-loop corrections to the magnon rapidities. The

asymptotic all-loop Baxter equation for twist-two operators and its perturbative solution, $P_M(u) = P_M^{(0)}(u) + g^2 P_M^{(2)}(u) + \dots$, have been studied in [42]. In particular, the two-loop solution $P_M^{(2)}(u)$ has been derived

$$\begin{aligned}
P_M^{(2)}(u) = & 4 \left(S_2(M) + 4 S_1(M)^2 - 2 S_1(M) S_1(2M) \right) {}_3F_2 \left(\begin{matrix} -M, & M+1, & \frac{1}{2} + iu \\ & 1, & 1 \end{matrix} \middle| 1 \right) \\
& + 4 S_1(M) \frac{\partial}{\partial \delta} {}_3F_2 \left(\begin{matrix} -M, & M+1+2\delta, & \frac{1}{2} + iu \\ & 1+\delta, & 1 \end{matrix} \middle| 1 \right) \Big|_{\delta=0} \\
& - \frac{\partial^2}{\partial \delta^2} {}_3F_2 \left(\begin{matrix} -M, & M+1, & \frac{1}{2} + iu \\ & 1+\delta, & 1-\delta \end{matrix} \middle| 1 \right) \Big|_{\delta=0} \\
& - \frac{\partial^2}{\partial \delta^2} {}_3F_2 \left(\begin{matrix} -M, & M+1, & \frac{1}{2} + iu + \delta \\ & 1+\delta, & 1+\delta \end{matrix} \middle| 1 \right) \Big|_{\delta=0}. \tag{5.25}
\end{aligned}$$

Knowing $P_M^{(2)}(u)$, it is straightforward to determine $Y^{(8,2)}(M, q, Q)$

$$\frac{Y_Q^{(8,2)}(M, q, Q)}{Y_Q^{(8,0)}(M, q, Q)} = 2 \frac{P_M^{(2)'}(-\frac{i}{2}) - (-1)^M P_M^{(2)'}(\frac{i}{2})}{P_M^{(0)'}(-\frac{i}{2}) - (-1)^M P_M^{(0)'}(\frac{i}{2})} - \frac{R_M^{(2)}(q, Q)}{R_M^{(0)}(q, Q)} + 2 \frac{T_M^{(2)}(q, Q)}{T_M^{(0)}(q, Q)}. \tag{5.26}$$

Here,

$$\frac{R_M^{(2)}(q, Q)}{R_M^{(0)}(q, Q)} = \frac{P_M^{(2)}(\frac{q-i(-1+Q)}{2})}{P_M^{(0)}(\frac{q-i(-1+Q)}{2})} + \frac{P_M^{(2)}(\frac{q+i(-1+Q)}{2})}{P_M^{(0)}(\frac{q+i(-1+Q)}{2})} + \frac{P_M^{(2)}(\frac{q-i(1+Q)}{2})}{P_M^{(0)}(\frac{q-i(1+Q)}{2})} + \frac{P_M^{(2)}(\frac{q+i(1+Q)}{2})}{P_M^{(0)}(\frac{q+i(1+Q)}{2})}, \tag{5.27}$$

and

$$T_M^{(2)}(q, Q) = \sum_{j=0}^{Q-1} \left(\frac{1}{2j - iq - Q} - (-1)^M \frac{1}{2(j+1) - iq - Q} \right) P_M^{(2)}\left(\frac{q - i(Q-1)}{2} + ij\right). \tag{5.28}$$

5.4 Modification of the asymptotic Bethe ansatz

An important correction that needs to be taken into consideration at the five-loop order is the modification of the quantisation condition. The form of this correction for an arbitrary number of magnons has been proposed in [19]. The rapidities of the individual magnons are influenced by the finite-size effects

$$u_j \rightarrow u_j + g^8 \delta u_j + \mathcal{O}(g^{10}). \tag{5.29}$$

This produces an additional contribution to the scaling dimension

$$\Delta_w^{\text{ABA}} = g^{10} \sum_{j=1}^M E'(u_j) \delta u_j + \mathcal{O}(g^{12}). \tag{5.30}$$

Following the arguments presented in [24], we expect the shifts δu_j to be determined through

$$\sum_{j=1}^M \left(\frac{\partial BY(u_k)}{\partial u_j} \right)_{u=u^0} \delta u_k + \Phi_k|_{u=u^0} = 0, \quad k = 1, \dots, M, \quad (5.31)$$

with

$$BY(u_k) = \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^2 \prod_{j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (5.32)$$

The one-loop Bethe roots u_k^0 are zeros of the function $(BY(u_k) - 1)$. The quantities Φ_k entering (5.31) are given by

$$\Phi_k = \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi i} \left(\frac{z^-}{z^+} \right)^2 \text{str} \left\{ S_Q(q, u_1) \dots \partial S_Q(q, u_k) \dots S_Q(q, u_M) \right\}. \quad (5.33)$$

In practice, due to

$$\Phi_k = (\Phi_1)_{u_1 \leftrightarrow u_k} \quad (5.34)$$

it is sufficient to calculate say Φ_1 . Using the $G(q, Q)$ matrix introduced in the sub-section 5.2, one finds for $k = 1$

$$\begin{aligned} \text{str} \left\{ (\partial S_Q) S_Q \dots S_Q \right\} &= \frac{\partial S_0(q, Q, u_1)}{\partial q} \left(\prod_{k=2}^M S_0(q, Q, u_k) \right) \text{str} \left\{ S_Q^{\text{diag}} \dots S_Q^{\text{diag}} \right\} \\ &+ \left(\prod_{k=1}^M S_0(q, Q, u_k) \right) \text{str} \left\{ (G^2 \partial S_Q^{\text{matrix}} (G^{-1})^2) S_Q^{\text{diag}} \dots S_Q^{\text{diag}} \right\}. \end{aligned} \quad (5.35)$$

Since all the matrices to the right of $G \partial S_Q^{\text{matrix}} G^{-1}$ are diagonal, only diagonal elements of $G \partial S_Q^{\text{matrix}} G^{-1}$ contribute to the supertrace. It should be noted that due to the tensor decomposition (5.17) one has

$$\begin{aligned} G^2 \partial S_Q^{\text{matrix}} (G^{-1})^2 &= (G \partial \hat{S}_{\text{su}(2|2), Q}(q, u_k) G^{-1}) \otimes \hat{S}_{\text{su}(2|2), Q}^{\text{diag}}(q, u_k) \\ &+ \hat{S}_{\text{su}(2|2), Q}^{\text{diag}}(q, u_k) \otimes (G \partial \hat{S}_{\text{su}(2|2), Q}(q, u_k) G^{-1}). \end{aligned} \quad (5.36)$$

To calculate (5.33) to five-loop order, it is sufficient to expand all quantities in (5.35), with the exception of $(z^-/z^+)^2$, to the two-loop order. The diagonal part of $G \partial \hat{S}_{\text{su}(2|2), Q} G^{-1}$ admits a decomposition similar to (5.19) with the corresponding functions

$$DB_1(j, u_k) = DB_{1,0}(j, u_k) + g^2 DB_{1,2}(j, u_k) + \mathcal{O}(g^4), \quad (5.37)$$

$$DB_2(j, u_k) = DB_{2,0}(j, u_k) + g^2 DB_{2,2}(j, u_k) + \mathcal{O}(g^4), \quad (5.38)$$

$$DF(j, u_k) = DF_0(j, u_k) + g^2 DF_2(j, u_k) + \mathcal{O}(g^4). \quad (5.39)$$

The index j again takes the values $0, \dots, Q-1$. We present the explicit form of the expansion coefficients in Appendix C.

5.5 The final result

Having all ingredients, it is only a matter of computational effort to calculate (5.2) and (5.30). As discussed in [24], the rational integrals over q may be performed by taking the residue at $q = iQ$. The remaining residues, when combined together, should *cancel* upon summing over $Q = 1, \dots, \infty$. This is a very non-trivial occurrence which gives evidence that the μ -terms are absent in the weak-coupling limit⁷. There is also a contribution to the integrand that involves polygamma functions (5.14). They provide an additional source of infinitely many poles that need to be taken into account while calculating the integral! Bearing this in mind, we have used `Mathematica` to set up the rational integrands and took the residue at $q = iQ$, while the non-rational integrands were treated separately. Methods for performing sums over Q have been developed in [24] and can be also applied to the current case. For higher values of M , however, we have found it more efficient to perform high precision computations and to use EZ-Face [60] to determine the transcendental structure. One starts with $Q = 1$ and performs numerical integration over q and repeats that procedure for $Q = 2, 3, \dots$. The series is fast convergent and the number of steps depends on the desired accuracy. For the first dozen of M , we have performed calculations with very high accuracy and subsequently determined the transcendental structure (5.43) using EZ-Face. This allowed us to conjecture the form of the functions multiplying zeta functions. For higher values of M the non-rational part can be subtracted from the full result and it is sufficient to perform computations to the accuracy that allows to rationalise the results. In what follows we will discuss the intriguing structure of the wrapping correction.

We assume that the wrapping correction also preserves the reciprocity symmetry. This implies that a part of the wrapping correction may be found from the lower order results

$$\Delta_w = \frac{1}{2} \Delta_w^{(8)} \dot{\gamma}_2 + \frac{1}{2} \dot{\Delta}_w^{(8)} \gamma_2 + \mathcal{P}_{10}^w, \quad (5.40)$$

where γ_2 is the one-loop anomalous dimension and $\Delta_w^{(8)}$ is the four-loop wrapping correction found in [23]

$$\gamma_2 = \mathcal{P}_2 = 8 S_1 = 4 \mathbb{S}_1, \quad (5.41)$$

$$\begin{aligned} \Delta_w^{(8)} = \mathcal{P}_8^w = & -128 S_1^2 \left(5 \zeta(5) + 4 S_{-2} \zeta(3) + 2 S_5 - 2 S_{-5} + 4 S_{3,-2} - 4 S_{-2,-3} \right. \\ & \left. - 4 S_{4,1} + 8 S_{-2,-2,1} \right) = 2 \mathcal{P}_2^2 \left(-5 \zeta(5) + 2 \mathbb{S}_2 \zeta(3) + (\mathbb{S}_{2,1,2} - \mathbb{S}_{3,1,1}) \right). \end{aligned} \quad (5.42)$$

The remaining part \mathcal{P}_{10}^w should have the following transcendental structure

$$\mathcal{P}_{10}^w = \zeta(7) T_{\zeta(7)} + \zeta(3)^2 T_{\zeta(3)^2} + \zeta(5) T_{\zeta(5)} + \zeta(3) T_{\zeta(3)} + T_{\text{rational}}, \quad (5.43)$$

which is a plausible generalisation of the five-loop result for the Konishi operator, see [24]. Applying the principle of maximal transcendentality we conclude that the transcendentality of the components $T_{\zeta(7)}$, $T_{\zeta(3)^2}$, $T_{\zeta(5)}$, $T_{\zeta(3)}$, T_{rational} should be equal to 2, 3, 4, 6 and 9

⁷We have explicitly checked the cancelation of the remaining poles for the first few values of M .

respectively. The lowest-transcendentality functions, $T_{\zeta(7)}$ and $T_{\zeta(3)^2}$, may be deduced by inspecting first few values of M

$$T_{\zeta(7)} = 13440 S_1^2, \quad (5.44)$$

$$T_{\zeta(3)^2} = -1536 S_1^3. \quad (5.45)$$

The reconstruction of $T_{\zeta(5)}$ and $T_{\zeta(3)}$ requires approximately a dozen of values of (5.1). A careful analysis of their structure suggests that the five-loop wrapping correction should have the following structure

$$\mathcal{P}_{10}^w = 2 \mathcal{P}_2^2 \tilde{T} + 2 \mathcal{P}_2 \left(2 \mathcal{P}_4 + \frac{1}{16} \mathcal{P}_2^3 \right) \left(-5 \zeta(5) + 2 \mathbb{S}_2 \zeta(3) + (\mathbb{S}_{2,1,2} - \mathbb{S}_{3,1,1}) \right), \quad (5.46)$$

$$\tilde{T} = \zeta(7) \tilde{T}_{\zeta(7)} + \zeta(3)^2 \tilde{T}_{\zeta(3)^2} + \zeta(5) \tilde{T}_{\zeta(5)} + \zeta(3) \tilde{T}_{\zeta(3)} + \tilde{T}_{\text{rational}}, \quad (5.47)$$

with \mathcal{P}_4 (4.10) and \tilde{T}_i given by

$$\mathcal{P}_4 = 8(\mathbb{S}_1 \mathbb{S}_2 - \mathbb{S}_{2,1} - \mathbb{S}_3), \quad (5.48)$$

$$\tilde{T}_{\zeta(7)} = 105, \quad (5.49)$$

$$\tilde{T}_{\zeta(3)^2} = -6 \mathbb{S}_1, \quad (5.50)$$

$$\tilde{T}_{\zeta(5)} = -40 \mathbb{S}_2, \quad (5.51)$$

$$\tilde{T}_{\zeta(3)} = 4(3 \mathbb{S}_1 \mathbb{S}_{2,1} - 2 \mathbb{S}_{2,2} + 2 \mathbb{S}_{3,1} - \mathbb{S}_{2,1,1} - \mathbb{S}_4). \quad (5.52)$$

The simplest ansatz for $\tilde{T}_{\text{rational}}$ requires 48 binomial sums. Thus, one needs to calculate (5.1) up to $M = 48$, which turns out to be very elaborate. The most complicated part of the calculations is the determination of the matrix part of the integrand and the calculation of the modification of the asymptotic Bethe ansatz. For example, for $M = 48$ this amounts to, respectively, 100 and 30 hours of computer time on a 2200 MHz PC.

At the end we have found the following result:

$$\begin{aligned} \tilde{T}_{\text{rational}} = & 2 \left(\mathbb{S}_1 (\mathbb{S}_{2,3,1} - \mathbb{S}_{3,1,2}) - \mathbb{S}_{2,1,4} + 2 \mathbb{S}_{2,2,3} - 5 \mathbb{S}_{3,1,3} + 2 \mathbb{S}_{3,2,2} \right. \\ & + 2 \mathbb{S}_{3,3,1} - \mathbb{S}_{4,1,2} + \mathbb{S}_{5,1,1} - 2 \mathbb{S}_{2,1,2,2} + 2 \mathbb{S}_{2,1,3,1} - 2 \mathbb{S}_{2,2,1,2} - 2 \mathbb{S}_{2,2,2,1} \\ & \left. + 2 \mathbb{S}_{2,3,1,1} - 2 \mathbb{S}_{3,1,1,2} + 2 \mathbb{S}_{3,1,2,1} + 2 \mathbb{S}_{3,2,1,1} - \mathbb{S}_{2,1,1,1,2} + \mathbb{S}_{3,1,1,1,1} \right). \end{aligned} \quad (5.53)$$

Putting (5.47)-(5.53) into equation (5.46) and upon expressing the full wrapping contribution (5.40) in terms of the nested harmonic sums, one finds (1.2).

It is interesting to note, as follows from comparing (5.46) and (5.42), that the five-loop function \mathcal{P}_{10}^w contains a part proportional to its four-loop counterpart \mathcal{P}_8^w with the prefactor re-scaled as $\mathcal{P}_2^2 \rightarrow \mathcal{P}_2 (2 \mathcal{P}_4 + \frac{1}{16} \mathcal{P}_2^3)$. Thus, remarkably, the one-loop function \mathcal{P}_2 is (up to the \mathcal{P}_2^3 term) simply replaced by the two-loop contribution \mathcal{P}_4 . The remaining part, on the other hand, is proportional to $\mathcal{P}_2^2 \sim \mathbb{S}_1^2$, similarly to the four-loop wrapping correction. If a similar decomposition is also a feature of higher orders of perturbation theory, this might considerably simplify the calculation of the wrapping corrections for twist-two operators.

5.6 Large M asymptotic and analytical continuation

In this section we check our result against the known constraints on the five-loop anomalous dimension of twist-two operators. Firstly, we find that the $M \rightarrow \infty$ limit can be easily calculated with use of the **SUMMER** package [40] for **FORM** [62] giving

$$\lim_{M \rightarrow \infty} \Delta_w(M) = 0. \quad (5.54)$$

This means that, in similarity to the four-loop case, the wrapping effects do not influence the scaling function and may contribute starting from the order $\mathcal{O}\left(\frac{\log M}{M}\right)$ only.

Secondly, we use the **SUMMER** package to transform the ABA contribution and the wrapping correction into the canonical basis. The harmonic sums corresponding to the highest poles at $M = -2 + \omega$ enter with the following coefficients

$$\begin{aligned} \gamma_{10}^{ABA} &= 1024 (13 S_9 + 15 S_{-9}) + \dots, \\ \Delta_w &= 1024 (S_9 - S_{-9}) + \dots \end{aligned} \quad (5.55)$$

Using the following analytical continuation

$$S_9(-2 + \omega) = \frac{1}{\omega^9}, \quad S_{-9}(-2 + \omega) = \frac{1}{\omega^9}, \quad (5.56)$$

we confirm the result (3.8) following from the double-logarithmic constraints.

The most intriguing test, however, is the comparison with the predictions coming from the BFKL equation, *cf.* (3.7). The expansion at $M = -1 + \omega$ of the five-loop ABA result was determined in (4.20)

$$\begin{aligned} \gamma_{10}^{ABA}(-1 + \omega) &= -\frac{2048}{\omega^9} + \frac{7168 \pi^2}{3 \omega^7} - \frac{5120 \zeta(3)}{\omega^6} - \frac{63872 \pi^4}{45 \omega^5} \\ &\quad + 512 \frac{13 \pi^2 \zeta(3) - 678 \zeta(5)}{3 \omega^4} + \mathcal{O}\left(\frac{1}{\omega^3}\right). \end{aligned} \quad (5.57)$$

For the five-loop wrapping correction (1.2) we find

$$\begin{aligned} \Delta_w(-1 + \omega) &= +\frac{2048}{\omega^9} - \frac{7168 \pi^2}{3 \omega^7} + \frac{5120 \zeta(3)}{\omega^6} + \frac{63872 \pi^4}{45 \omega^5} \\ &\quad - 512 \frac{5 \pi^2 \zeta(3) - 194 \zeta(5)}{3 \omega^4} + \mathcal{O}\left(\frac{1}{\omega^3}\right). \end{aligned} \quad (5.58)$$

Summing up these two results gives

$$\gamma_{10}(-1 + \omega) = -1024 \frac{2 \zeta(2) \zeta(3) + 16 \zeta(5)}{\omega^4} + \mathcal{O}\left(\frac{1}{\omega^3}\right), \quad (5.59)$$

which coincides with the BFKL equation prediction (3.7)! The full agreement with the known constraints strongly corroborates our result!

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A The ABA contribution at five-loop order

Below we present an explicit expression for the five-loop ABA contribution to the anomalous dimension of twist-two operators γ_{10}^{ABA} in terms of the usual harmonic sums (2.4). We *do not* use the canonical basis, see section 2 for the definition, since this would expand the result even further.

$$\begin{aligned}
& (20480S_{-5} - 8192S_{-3}S_{-2} + 2048S_5 - 20480S_{-4,1} - 16384S_{-3,2} - \frac{28672}{3}S_{-2,3} \\
& + \frac{32768}{3}S_{-3,1,1} + \frac{16384}{3}S_{-2,1,2} + \frac{16384}{3}S_{-2,2,1})S_1^4 + (20480S_{-3}^2 + 4096S_3^2 + 81920S_{-6} \\
& + S_{-2}(30720S_{-4} + 8192S_4) + 30720S_6 - 98304S_{-5,1} - 12288S_{-4,-2} - 102400S_{-4,2} \\
& - 8192S_{-3,-3} - 90112S_{-3,3} + S_3(24576S_{-3} - 16384S_{-2,1}) - 57344S_{-2,4} + 4096S_{4,2} \\
& + 16384S_{5,1} + 122880S_{-4,1,1} - 16384S_{-3,-2,1} + 106496S_{-3,1,2} + 106496S_{-3,2,1} \\
& - 16384S_{-2,-3,1} - 8192S_{-2,-2,2} + S_2(-8192S_{-2}^2 + 49152S_{-4} + 8192S_4 - \frac{131072}{3}S_{-3,1} \\
& - \frac{81920}{3}S_{-2,2} + \frac{65536}{3}S_{-2,1,1}) + 65536S_{-2,1,3} + 65536S_{-2,2,2} + 65536S_{-2,3,1} \\
& - 98304S_{-3,1,1,1} - 49152S_{-2,1,1,2} - 49152S_{-2,1,2,1} - 49152S_{-2,2,1,1})S_1^3 + ((12288S_{-3} \\
& + 9216S_3)S_{-2}^2 + (53248S_{-5} + 24576S_5 - 61440S_{-4,1} - 40960S_{-3,2} - 20480S_{-2,3} \\
& + 32768S_{-3,1,1} + 16384S_{-2,1,2} + 16384S_{-2,2,1})S_{-2} + 113664S_{-7} + 3072S_7 - 163840S_{-6,1} \\
& - 172032S_{-5,2} - 174080S_{-4,3} - 163840S_{-3,4} + S_2^2(36864S_{-3} + 12288S_3 - 24576S_{-2,1}) \\
& + (-12288S_{-4} - 36864S_4)S_{-2,1} - 118784S_{-2,5} + 8192S_{4,3} + 8192S_{5,2} - 40960S_{6,1} \\
& + 253952S_{-5,1,1} + 24576S_{-4,-2,1} + 24576S_{-4,1,-2} + 266240S_{-4,1,2} + 266240S_{-4,2,1} \\
& + 16384S_{-3,-3,1} - 8192S_{-3,-2,2} + 16384S_{-3,1,-3} + 249856S_{-3,1,3} + 8192S_{-3,2,-2} \\
& + 258048S_{-3,2,2} + 249856S_{-3,3,1} - 16384S_{-2,-3,2} - 16384S_{-2,-2,3} + S_{-3}(14336S_{-4} \\
& + 43008S_4 - 49152S_{-3,1} - 24576S_{-2,2} + 32768S_{-2,1,1}) + S_3(52224S_{-4} + 12288S_4 \\
& - 57344S_{-3,1} - 40960S_{-2,2} + 49152S_{-2,1,1}) + 172032S_{-2,1,4} + 180224S_{-2,2,3} \\
& + 180224S_{-2,3,2} + 172032S_{-2,4,1} - 8192S_{4,1,2} - 8192S_{4,2,1} - 32768S_{5,1,1} \\
& - 368640S_{-4,1,1,1} + 32768S_{-3,-2,1,1} - 344064S_{-3,1,1,2} - 344064S_{-3,1,2,1} - 344064S_{-3,2,1,1} \\
& + 32768S_{-2,-3,1,1} + 16384S_{-2,-2,1,2} + 16384S_{-2,-2,2,1} + S_2(92160S_{-5} + S_{-2}(49152S_{-3} \\
& + 24576S_3) + 30720S_5 - 122880S_{-4,1} - 12288S_{-3,-2} - 122880S_{-3,2} - 86016S_{-2,3} \\
& + 12288S_{4,1} + 172032S_{-3,1,1} - 24576S_{-2,-2,1} + 122880S_{-2,1,2} + 122880S_{-2,2,1} \\
& - 147456S_{-2,1,1,1}) - 221184S_{-2,1,1,3} - 221184S_{-2,1,2,2} - 221184S_{-2,1,3,1} \\
& - 221184S_{-2,2,1,2} - 221184S_{-2,2,2,1} - 221184S_{-2,3,1,1} + 393216S_{-3,1,1,1,1} \\
& + 196608S_{-2,1,1,1,2} + 196608S_{-2,1,1,2,1} + 196608S_{-2,1,2,1,1} + 196608S_{-2,2,1,1,1})S_1^2 \\
& + (2048S_2^4 + 8192S_{-2}S_2^3 + (9216S_{-2}^2 + 24576S_{-4} + 9216S_4 - 36864S_{-3,1} - 30720S_{-2,2} \\
& + 49152S_{-2,1,1})S_2^2 + (4096S_{-2}^3 + (32768S_{-4} + 24576S_4 - 49152S_{-3,1} - 24576S_{-2,2} \\
& + 32768S_{-2,1,1})S_{-2} + 6144S_3^2 + 53248S_{-6} + 6144S_6 - 90112S_{-5,1} - 94208S_{-4,2} \\
& - 94208S_{-3,3} + S_3(32768S_{-3} - 32768S_{-2,1}) - 16384S_{-3}S_{-2,1} - 77824S_{-2,4} + 8192S_{4,2} \\
& - 16384S_{5,1} + 163840S_{-4,1,1} + 16384S_{-3,-2,1} + 16384S_{-3,1,-2} + 172032S_{-3,1,2} \\
& + 172032S_{-3,2,1} - 16384S_{-2,-2,2} + 139264S_{-2,1,3} + 147456S_{-2,2,2} + 139264S_{-2,3,1}
\end{aligned}$$

$$\begin{aligned}
& -16384S_{4,1,1} - 294912S_{-3,1,1,1} + 32768S_{-2,-2,1,1} - 245760S_{-2,1,1,2} - 245760S_{-2,1,2,1} \\
& - 245760S_{-2,2,1,1} + 393216S_{-2,1,1,1,1})S_2 + 13824S_{-4}^2 + 4608S_4^2 + 16384S_{-3,1}^2 \\
& + 14336S_{-2,2}^2 + 57344S_{-8} + S_{-2}^2(3072S_{-4} + 12288S_4) + 64512S_8 - 98304S_{-7,1} \\
& - 30720S_{-6,-2} - 98304S_{-6,2} - 16384S_{-5,-3} - 102400S_{-5,3} - 3072S_{-4,-4} - 98304S_{-4,4} \\
& - 98304S_{-3,5} - 92160S_{-2,6} - 15360S_{4,4} - 12288S_{5,3} + 26624S_{6,2} + 36864S_{7,1} \\
& + 163840S_{-6,1,1} - 24576S_{-5,-2,1} + 180224S_{-5,1,2} + 180224S_{-5,2,1} - 24576S_{-4,-3,1} \\
& - 6144S_{-4,-2,-2} - 18432S_{-4,-2,2} + 184320S_{-4,1,3} + 196608S_{-4,2,2} + 184320S_{-4,3,1} \\
& - 8192S_{-3,-4,1} - 4096S_{-3,-3,-2} - 28672S_{-3,-3,2} - 4096S_{-3,-2,-3} + 12288S_{-3,-2,3} \\
& + 180224S_{-3,1,4} + 192512S_{-3,2,3} + 192512S_{-3,3,2} + 176128S_{-3,4,1} + 8192S_{-2,-5,1} \\
& - 22528S_{-2,-4,2} + 4096S_{-2,-3,3} + 30720S_{-2,-2,4} + S_{-3,1}(36864S_{-2,2} - 16384S_{-2,1,1}) \\
& - 8192S_{-2,2}S_{-2,1,1} + S_{-4}(-14336S_{-3,1} - 10240S_{-2,2} + 36864S_{-2,1,1}) \\
& + S_4(30720S_{-4} - 51200S_{-3,1} - 43008S_{-2,2} + 69632S_{-2,1,1}) + 139264S_{-2,1,5} \\
& + S_{-2,1}(-4096S_{-5} - 20480S_5 + 24576S_{-4,1} + 36864S_{-3,2} + 28672S_{-2,3} - 16384S_{-3,1,1} \\
& - 8192S_{-2,1,2} - 8192S_{-2,2,1}) + 145408S_{-2,2,4} + 147456S_{-2,3,3} + 143360S_{-2,4,2} \\
& + 131072S_{-2,5,1} - 8192S_{4,1,3} - 8192S_{4,2,2} - 8192S_{4,3,1} - 16384S_{5,1,2} - 16384S_{5,2,1} \\
& - 294912S_{-5,1,1,1} - 319488S_{-4,1,1,2} - 319488S_{-4,1,2,1} - 319488S_{-4,2,1,1} + 49152S_{-3,-3,1,1} \\
& + 8192S_{-3,-2,-2,1} + 16384S_{-3,-2,1,2} + 16384S_{-3,-2,2,1} - 16384S_{-3,1,1,-3} - 311296S_{-3,1,1,3} \\
& - 327680S_{-3,1,2,2} - 311296S_{-3,1,3,1} - 16384S_{-3,2,-2,1} - 327680S_{-3,2,1,2} - 327680S_{-3,2,2,1} \\
& - 311296S_{-3,3,1,1} + 73728S_{-2,-4,1,1} + 8192S_{-2,-3,-2,1} + 40960S_{-2,-3,1,2} + 40960S_{-2,-3,2,1} \\
& + 8192S_{-2,-2,-3,1} + 4096S_{-2,-2,-2,2} + 16384S_{-2,-2,1,3} + 16384S_{-2,-2,2,2} + 16384S_{-2,-2,3,1} \\
& - 24576S_{-2,1,1,-4} + S_{-3}(40960S_{-5} + 16384S_5 - 28672S_{-4,1} - 22528S_{-3,2} - 22528S_{-2,3} \\
& + 4096S_{4,1} + 49152S_{-3,1,1} - 8192S_{-2,-2,1} + 36864S_{-2,1,2} + 36864S_{-2,2,1} \\
& - 49152S_{-2,1,1,1}) + S_3(40960S_{-5} + 8192S_5 - 53248S_{-4,1} - 51200S_{-3,2} - \frac{112640S_{-2,3}}{3} \\
& + \frac{212992}{3}S_{-3,1,1} + \frac{143360}{3}S_{-2,1,2} + \frac{143360}{3}S_{-2,2,1} - 49152S_{-2,1,1,1}) - 221184S_{-2,1,1,4} \\
& - 8192S_{-2,1,2,-3} - 237568S_{-2,1,2,3} - 237568S_{-2,1,3,2} - 221184S_{-2,1,4,1} - 16384S_{-2,2,-3,1} \\
& - 8192S_{-2,2,-2,2} - 8192S_{-2,2,1,-3} + S_{-2}(4096S_{-3}^2 + 8192S_3^2 + 56320S_{-6} + 25600S_6 \\
& - 32768S_{-5,1} - 26624S_{-4,2} - 28672S_{-3,3} + S_3(20480S_{-3} - 8192S_{-2,1}) - 24576S_{-2,4} \\
& + 2048S_{4,2} + 8192S_{5,1} + 36864S_{-4,1,1} - 8192S_{-3,-2,1} + 36864S_{-3,1,2} + 36864S_{-3,2,1} \\
& - 8192S_{-2,-3,1} - 4096S_{-2,-2,2} + 24576S_{-2,1,3} + 24576S_{-2,2,2} + 24576S_{-2,3,1} \\
& - 49152S_{-3,1,1,1} - 24576S_{-2,1,1,2} - 24576S_{-2,1,2,1} - 24576S_{-2,2,1,1}) - 237568S_{-2,2,1,3} \\
& - 245760S_{-2,2,2,2} - 237568S_{-2,2,3,1} - 16384S_{-2,3,-2,1} - 237568S_{-2,3,1,2} - 237568S_{-2,3,2,1} \\
& - 221184S_{-2,4,1,1} + 24576S_{4,1,1,2} + 24576S_{4,1,2,1} + 24576S_{4,2,1,1} + 98304S_{5,1,1,1}
\end{aligned}$$

$$\begin{aligned}
& +491520S_{-4,1,1,1,1} - 98304S_{-3,-2,1,1,1} - 32768S_{-3,1,-2,1,1} + 491520S_{-3,1,1,1,2} \\
& +491520S_{-3,1,1,2,1} + 491520S_{-3,1,2,1,1} + 491520S_{-3,2,1,1,1} - 98304S_{-2,-3,1,1,1} \\
& -49152S_{-2,-2,1,1,2} - 49152S_{-2,-2,1,2,1} - 49152S_{-2,-2,2,1,1} - 32768S_{-2,1,-3,1,1} \\
& -16384S_{-2,1,-2,1,2} - 16384S_{-2,1,-2,2,1} + 327680S_{-2,1,1,1,3} + 327680S_{-2,1,1,2,2} \\
& +327680S_{-2,1,1,3,1} + 327680S_{-2,1,2,1,2} + 327680S_{-2,1,2,2,1} + 327680S_{-2,1,3,1,1} \\
& -16384S_{-2,2,-2,1,1} + 327680S_{-2,2,1,1,2} + 327680S_{-2,2,1,2,1} + 327680S_{-2,2,2,1,1} \\
& +327680S_{-2,3,1,1,1} - 655360S_{-3,1,1,1,1,1} - 327680S_{-2,1,1,1,1,2} - 327680S_{-2,1,1,1,2,1} \\
& -327680S_{-2,1,1,2,1,1} - 327680S_{-2,1,2,1,1,1} - 327680S_{-2,2,1,1,1,1})S_1 + 512S_3^3 - 7168S_{-9} \\
& +7168S_9 - 18432S_{-8,1} - 2048S_{-2,-7} + S_3^2(3072S_{-3} - 2048S_{-2,1}) + S_2^3(1024S_{-3} \\
& +1024S_3 - 2048S_{-2,1}) + S_{-2}(3072S_{-3}S_4 - 6144S_{-2,1}S_4 + S_3(3072S_{-4} + 6144S_4 \\
& -4096S_{-3,1} - 2048S_{-2,2})) - 8192S_{1,-8} + 8192S_{1,8} - 16384S_{2,-7} + 16384S_{2,7} \\
& -3072S_{3,-6} + 3072S_{3,6} - 13824S_{4,-5} + 4608S_{4,5} - 34816S_{5,-4} - 2048S_{5,4} - 35328S_{6,-3} \\
& -4608S_{6,3} + 10240S_{7,-2} + 9216S_{7,2} + 16384S_{8,1} + 26624S_{-7,1,1} - 27648S_{-6,-2,1} \\
& -6144S_{-6,1,-2} + 12288S_{-6,1,2} + 12288S_{-6,2,1} - 18432S_{-5,-3,1} - 2048S_{-5,-2,-2} \\
& -4096S_{-5,-2,2} - 18432S_{-5,1,-3} - 4096S_{-5,2,-2} + 26624S_{-4,-4,1} + 44032S_{-4,-3,-2} \\
& +51200S_{-4,-3,2} + 70656S_{-4,-2,-3} + 12288S_{-4,-2,3} + 13312S_{-4,1,-4} + 17408S_{-4,1,4} \\
& +7168S_{-4,2,-3} - 1024S_{-4,3,-2} + 44032S_{-4,4,1} - 10240S_{-3,-5,1} + 45056S_{-3,-4,-2} \\
& +51200S_{-3,-4,2} + 157696S_{-3,-3,-3} + 33792S_{-3,-3,3} + 73728S_{-3,-2,-4} + 8192S_{-3,-2,4} \\
& -8192S_{-3,1,-5} + 61440S_{-3,1,5} + 14336S_{-3,2,-4} + 20480S_{-3,2,4} - 3072S_{-3,3,-3} \\
& +10240S_{-3,4,-2} + 45056S_{-3,4,2} + 90112S_{-3,5,1} - 13312S_{-2,-6,1} + 1024S_{-2,-5,-2} \\
& -4096S_{-2,-5,2} + 68608S_{-2,-4,-3} + 12288S_{-2,-4,3} + 70656S_{-2,-3,-4} + 8192S_{-2,-3,4} \\
& +15360S_{-2,-2,-5} + 7168S_{-2,-2,5} - 7168S_{-2,1,-6} + 21504S_{-2,1,6} - 10240S_{-7,-2} \\
& -13312S_{-7,2} + 16896S_{-6,-3} - 5632S_{-6,3} + 5120S_{-5,-4} + 1024S_{-5,4} + 3584S_{-4,-5} \\
& -27136S_{-4,5} + 9216S_{-3,-6} - 23552S_{-3,6} - 4096S_{-2,2,-5} + 28672S_{-2,2,5} \\
& +1024S_{-2,3,4} + 8192S_{-2,4,-3} + 11264S_{-2,4,3} + 13312S_{-2,5,-2} + 40960S_{-2,5,2} \\
& +35840S_{-2,6,1} + 40960S_{1,-7,1} - 11264S_{1,-6,-2} + 8192S_{1,-6,2} - 32768S_{1,-5,-3} \\
& +4096S_{1,-5,3} + 18432S_{1,-4,-4} + 23552S_{1,-4,4} - 10240S_{1,-3,-5} + 71680S_{1,-3,5} \\
& -11264S_{1,-2,-6} + 25600S_{1,-2,6} + 32768S_{1,1,-7} - 32768S_{1,1,7} + 8192S_{1,2,-6} - 8192S_{1,2,6} \\
& +4096S_{1,3,-5} + 35840S_{1,4,-4} - 6144S_{1,4,4} + 83968S_{1,5,-3} + 18432S_{1,5,3} + 17408S_{1,6,-2} \\
& +22528S_{1,6,2} - 32768S_{1,7,1} + 14336S_{2,-6,1} - 20480S_{2,-5,-2} - 8192S_{2,-5,2} \\
& +22528S_{2,-4,-3} + 1024S_{2,-4,3} + 32768S_{2,-3,-4} + 30720S_{2,-3,4} - 6144S_{2,-2,-5} \\
& +38912S_{2,-2,5} + 8192S_{2,1,-6} - 8192S_{2,1,6} - 4096S_{2,2,-5} + 16384S_{2,2,5} - 1024S_{2,3,-4} \\
& -5120S_{2,3,4} + 43008S_{2,4,-3} + 9216S_{2,4,3} + 32768S_{2,5,-2} + 40960S_{2,5,2} + 6144S_{2,6,1} \\
& +2048S_{3,-5,1} - 3072S_{3,-4,-2} - 3072S_{3,-4,2} + 12288S_{3,-3,-3} + 1024S_{3,-3,3} + 5120S_{3,-2,-4}
\end{aligned}$$

$$\begin{aligned}
& +7168S_{3,-2,4} + 4096S_{3,1,-5} - 1024S_{3,2,-4} - 5120S_{3,2,4} + 3072S_{3,3,-3} + 9216S_{3,4,-2} \\
& + 9216S_{3,4,2} + 8192S_{3,5,1} + 39936S_{4,-4,1} - 6144S_{4,-3,-2} + 31744S_{4,-3,2} - 6144S_{4,-2,-3} \\
& + 15360S_{4,-2,3} + 32768S_{4,1,-4} - 6144S_{4,1,4} + 36864S_{4,2,-3} + 9216S_{4,2,3} + 8192S_{4,3,-2} \\
& + 9216S_{4,3,2} - 6144S_{4,4,1} + 86016S_{5,-3,1} + 8192S_{5,-2,-2} + 36864S_{5,-2,2} + 81920S_{5,1,-3} \\
& + 18432S_{5,1,3} + 32768S_{5,2,-2} + 40960S_{5,2,2} + 18432S_{5,3,1} + 50176S_{6,-2,1} + 20480S_{6,1,-2} \\
& + 22528S_{6,1,2} + 22528S_{6,2,1} - 18432S_{7,1,1} - 24576S_{-6,1,1,1} + 8192S_{-5,-2,1,1} \\
& + 28672S_{-5,1,-2,1} + 8192S_{-5,1,1,-2} - 102400S_{-4,-3,1,1} - 88064S_{-4,-2,-2,1} \\
& - 53248S_{-4,-2,1,-2} - 59392S_{-4,-2,1,2} - 59392S_{-4,-2,2,1} - 55296S_{-4,1,-3,1} \\
& - 34816S_{-4,1,-2,-2} - 43008S_{-4,1,-2,2} - 14336S_{-4,1,1,-3} - 2048S_{-4,1,2,-2} - 12288S_{-4,2,-2,1} \\
& - 2048S_{-4,2,1,-2} - 102400S_{-3,-4,1,1} - 188416S_{-3,-3,-2,1} - 126976S_{-3,-3,1,-2} \\
& - 155648S_{-3,-3,1,2} - 155648S_{-3,-3,2,1} - 180224S_{-3,-2,-3,1} - 24576S_{-3,-2,-2,-2} \\
& - 90112S_{-3,-2,-2,2} - 155648S_{-3,-2,1,-3} - 36864S_{-3,-2,1,3} - 65536S_{-3,-2,2,-2} \\
& - 81920S_{-3,-2,2,2} - 36864S_{-3,-2,3,1} - 61440S_{-3,1,-4,1} - 102400S_{-3,1,-3,-2} \\
& - 122880S_{-3,1,-3,2} - 159744S_{-3,1,-2,-3} - 30720S_{-3,1,-2,3} - 28672S_{-3,1,1,-4} \\
& - 40960S_{-3,1,1,4} - 12288S_{-3,1,2,-3} + 2048S_{-3,1,3,-2} - 98304S_{-3,1,4,1} - 61440S_{-3,2,-3,1} \\
& - 40960S_{-3,2,-2,-2} - 49152S_{-3,2,-2,2} - 12288S_{-3,2,1,-3} + 4096S_{-3,3,-2,1} + 2048S_{-3,3,1,-2} \\
& - 90112S_{-3,4,1,1} + 8192S_{-2,-5,1,1} - 83968S_{-2,-4,-2,1} - 53248S_{-2,-4,1,-2} - 59392S_{-2,-4,1,2} \\
& - 59392S_{-2,-4,2,1} - 169984S_{-2,-3,-3,1} - 24576S_{-2,-3,-2,-2} - 83968S_{-2,-3,-2,2} \\
& - 151552S_{-2,-3,1,-3} - 36864S_{-2,-3,1,3} - 65536S_{-2,-3,2,-2} - 81920S_{-2,-3,2,2} \\
& - 36864S_{-2,-3,3,1} - 75776S_{-2,-2,-4,1} - 24576S_{-2,-2,-3,-2} - 79872S_{-2,-2,-3,2} \\
& - 24576S_{-2,-2,-2,-3} - 22528S_{-2,-2,-2,3} - 69632S_{-2,-2,1,-4} - 8192S_{-2,-2,1,4} \\
& - 73728S_{-2,-2,2,-3} - 18432S_{-2,-2,2,3} - 16384S_{-2,-2,3,-2} - 18432S_{-2,-2,3,2} \\
& - 8192S_{-2,-2,4,1} + 12288S_{-2,1,-5,1} - 38912S_{-2,1,-4,-2} - 43008S_{-2,1,-4,2} \\
& - 157696S_{-2,1,-3,-3} - 30720S_{-2,1,-3,3} - 71680S_{-2,1,-2,-4} - 8192S_{-2,1,-2,4} \\
& + 8192S_{-2,1,1,-5} + S_{-4}(4608S_{-5} + 1536S_5 - 9216S_{-4,1} - 9216S_{-3,2} - 9216S_{-2,3} \\
& + 18432S_{-3,1,1} + 18432S_{-2,1,2} + 18432S_{-2,2,1} - 36864S_{-2,1,1,1}) + S_4(4608S_{-5} + 1536S_5 \\
& - 9216S_{-4,1} - 9216S_{-3,2} - 9216S_{-2,3} + 18432S_{-3,1,1} + 18432S_{-2,1,2} + 18432S_{-2,2,1} \\
& - 36864S_{-2,1,1,1}) + S_2^2(3072S_{-5} + 1024S_5 - 6144S_{-4,1} - 6144S_{-3,2} + S_{-2}(2048S_{-3} \\
& + 4096S_3 - 4096S_{-2,1}) - 6144S_{-2,3} + 12288S_{-3,1,1} + 12288S_{-2,1,2} + 12288S_{-2,2,1} \\
& - 24576S_{-2,1,1,1}) + S_{-2,2}(-3072S_{-5} - 1024S_5 + 6144S_{-4,1} + 6144S_{-3,2} + 6144S_{-2,3} \\
& - 12288S_{-3,1,1} - 12288S_{-2,1,2} - 12288S_{-2,2,1} + 24576S_{-2,1,1,1}) + S_{-3,1}(-6144S_{-5} \\
& - 2048S_5 + 12288S_{-4,1} + 12288S_{-3,2} + 12288S_{-2,3} - 24576S_{-3,1,1} - 24576S_{-2,1,2} \\
& - 24576S_{-2,2,1} + 49152S_{-2,1,1,1}) - 57344S_{-2,1,1,5} - 8192S_{-2,1,2,-4} - 14336S_{-2,1,2,4} \\
& + 4096S_{-2,1,3,-3} - 12288S_{-2,1,4,-2} - 43008S_{-2,1,4,2} - 90112S_{-2,1,5,1} - 20480S_{-2,2,-4,1}
\end{aligned}$$

$$\begin{aligned}
& -43008S_{-2,2,-3,-2} - 49152S_{-2,2,-3,2} - 79872S_{-2,2,-2,-3} - 12288S_{-2,2,-2,3} \\
& -8192S_{-2,2,1,-4} + S_{-3}(7680S_{-6} + 2560S_6 - 12288S_{-5,1} - 12288S_{-4,2} - 12288S_{-3,3} \\
& -9216S_{-2,4} + 18432S_{-4,1,1} + 18432S_{-3,1,2} + 18432S_{-3,2,1} + 12288S_{-2,1,3} + 12288S_{-2,2,2} \\
& + 12288S_{-2,3,1} - 24576S_{-3,1,1,1} - 12288S_{-2,1,1,2} - 12288S_{-2,1,2,1} - 12288S_{-2,2,1,1}) \\
& + S_3(2560S_{-3}^2 - 6144S_{-2,1}S_{-3} + 2048S_{-2,1}^2 + 7680S_{-6} + 2560S_6 - 12288S_{-5,1} \\
& - 12288S_{-4,2} - 12288S_{-3,3} - 9216S_{-2,4} + 18432S_{-4,1,1} + 18432S_{-3,1,2} + 18432S_{-3,2,1} \\
& + 12288S_{-2,1,3} + 12288S_{-2,2,2} + 12288S_{-2,3,1} - 24576S_{-3,1,1,1} - 12288S_{-2,1,1,2} \\
& - 12288S_{-2,1,2,1} - 12288S_{-2,2,1,1}) + S_{-2,1}(-15360S_{-6} - 5120S_6 + 24576S_{-5,1} \\
& + 24576S_{-4,2} + 24576S_{-3,3} + 18432S_{-2,4} - 36864S_{-4,1,1} - 36864S_{-3,1,2} \\
& - 36864S_{-3,2,1} - 24576S_{-2,1,3} - 24576S_{-2,2,2} - 24576S_{-2,3,1} + 49152S_{-3,1,1,1} \\
& + 24576S_{-2,1,1,2} + 24576S_{-2,1,2,1} + 24576S_{-2,2,1,1}) - 14336S_{-2,2,1,4} \\
& - 51200S_{-2,2,4,1} + 2048S_{-2,3,-3,1} - 2048S_{-2,3,-2,-2} - 2048S_{-2,3,-2,2} + 4096S_{-2,3,1,-3} \\
& - 4096S_{-2,4,-2,1} - 12288S_{-2,4,1,-2} - 38912S_{-2,4,1,2} - 38912S_{-2,4,2,1} \\
& - 81920S_{-2,5,1,1} - 16384S_{1,-6,1,1} + 40960S_{1,-5,-2,1} + 24576S_{1,-5,1,-2} - 83968S_{1,-4,-3,1} \\
& - 51200S_{1,-4,-2,-2} - 59392S_{1,-4,-2,2} - 28672S_{1,-4,1,-3} + 2048S_{1,-4,1,3} - 4096S_{1,-4,2,-2} \\
& + 2048S_{1,-4,3,1} - 96256S_{1,-3,-4,1} - 129024S_{1,-3,-3,-2} - 155648S_{1,-3,-3,2} \\
& - 165888S_{1,-3,-2,-3} - 36864S_{1,-3,-2,3} - 51200S_{1,-3,1,-4} - 59392S_{1,-3,1,4} \\
& - 40960S_{1,-3,2,-3} + 8192S_{1,-3,3,-2} - 96256S_{1,-3,4,1} + 8192S_{1,-2,-5,1} - 51200S_{1,-2,-4,-2} \\
& - 59392S_{1,-2,-4,2} - 157696S_{1,-2,-3,-3} - 36864S_{1,-2,-3,3} - 75776S_{1,-2,-2,-4} \\
& - 8192S_{1,-2,-2,4} + 8192S_{1,-2,1,-5} - 65536S_{1,-2,1,5} - 20480S_{1,-2,2,-4} - 26624S_{1,-2,2,4} \\
& + 2048S_{1,-2,3,-3} - 8192S_{1,-2,4,-2} - 47104S_{1,-2,4,2} - 90112S_{1,-2,5,1} - 28672S_{1,1,-6,1} \\
& + 40960S_{1,1,-5,-2} + 16384S_{1,1,-5,2} - 45056S_{1,1,-4,-3} - 2048S_{1,1,-4,3} - 65536S_{1,1,-3,-4} \\
& - 61440S_{1,1,-3,4} + 12288S_{1,1,-2,-5} - 77824S_{1,1,-2,5} - 16384S_{1,1,1,-6} + 16384S_{1,1,1,6} \\
& + 8192S_{1,1,2,-5} - 32768S_{1,1,2,5} + 2048S_{1,1,3,-4} + 10240S_{1,1,3,4} - 86016S_{1,1,4,-3} \\
& - 18432S_{1,1,4,3} - 65536S_{1,1,5,-2} - 81920S_{1,1,5,2} - 12288S_{1,1,6,1} + 16384S_{1,2,-5,1} \\
& - 2048S_{1,2,-4,-2} + 4096S_{1,2,-4,2} - 110592S_{1,2,-3,-3} + 4096S_{1,2,-3,3} - 34816S_{1,2,-2,-4} \\
& - 28672S_{1,2,-2,4} + 8192S_{1,2,1,-5} - 32768S_{1,2,1,5} - 36864S_{1,2,4,-2} - 40960S_{1,2,4,2} \\
& - 65536S_{1,2,5,1} - 2048S_{1,3,-4,1} + 10240S_{1,3,-3,-2} + 4096S_{1,3,-3,2} - 14336S_{1,3,-2,-3} \\
& - 4096S_{1,3,-2,3} + 2048S_{1,3,1,-4} + 10240S_{1,3,1,4} - 14336S_{1,3,4,1} - 75776S_{1,4,-3,1} \\
& + 8192S_{1,4,-2,-2} - 30720S_{1,4,-2,2} - 77824S_{1,4,1,-3} - 18432S_{1,4,1,3} - 32768S_{1,4,2,-2} \\
& - 40960S_{1,4,2,2} - 18432S_{1,4,3,1} - 102400S_{1,5,-2,1} - 65536S_{1,5,1,-2} - 81920S_{1,5,1,2} \\
& - 81920S_{1,5,2,1} - 45056S_{1,6,1,1} + 16384S_{2,-5,1,1} - 38912S_{2,-4,-2,1} - 6144S_{2,-4,1,-2} \\
& - 4096S_{2,-4,1,2} - 4096S_{2,-4,2,1} - 139264S_{2,-3,-3,1} - 69632S_{2,-3,-2,-2} - 81920S_{2,-3,-2,2}
\end{aligned}$$

$$\begin{aligned}
& -73728S_{2,-3,1,-3} + 4096S_{2,-3,1,3} - 16384S_{2,-3,2,-2} + 4096S_{2,-3,3,1} - 55296S_{2,-2,-4,1} \\
& -69632S_{2,-2,-3,-2} - 81920S_{2,-2,-3,2} - 86016S_{2,-2,-2,-3} - 18432S_{2,-2,-2,3} \\
& -30720S_{2,-2,1,-4} - 32768S_{2,-2,1,4} - 28672S_{2,-2,2,-3} + 6144S_{2,-2,3,-2} - 49152S_{2,-2,4,1} \\
& +16384S_{2,1,-5,1} - 2048S_{2,1,-4,-2} + 4096S_{2,1,-4,2} - 110592S_{2,1,-3,-3} + 4096S_{2,1,-3,3} \\
& -34816S_{2,1,-2,-4} - 28672S_{2,1,-2,4} + 8192S_{2,1,1,-5} - 32768S_{2,1,1,5} - 36864S_{2,1,4,-2} \\
& -40960S_{2,1,4,2} - 65536S_{2,1,5,1} - 16384S_{2,2,-3,-2} - 8192S_{2,2,-3,2} - 65536S_{2,2,-2,-3} \\
& -49152S_{2,2,4,1} + 8192S_{2,3,-3,1} + 10240S_{2,3,-2,-2} + 8192S_{2,3,-2,2} - 49152S_{2,4,-2,1} \\
& -36864S_{2,4,1,-2} - 40960S_{2,4,1,2} - 40960S_{2,4,2,1} - 81920S_{2,5,1,1} + 6144S_{3,-4,1,1} \\
& -22528S_{3,-3,-2,1} - 2048S_{3,-3,1,-2} - 4096S_{3,-3,1,2} - 4096S_{3,-3,2,1} - 26624S_{3,-2,-3,1} \\
& -18432S_{3,-2,-2,-2} - 18432S_{3,-2,-2,2} - 10240S_{3,-2,1,-3} + 2048S_{3,-2,1,3} - 2048S_{3,-2,2,-2} \\
& +2048S_{3,-2,3,1} - 2048S_{3,1,-4,1} + 10240S_{3,1,-3,-2} + 4096S_{3,1,-3,2} - 14336S_{3,1,-2,-3} \\
& -4096S_{3,1,-2,3} + 2048S_{3,1,1,-4} + 10240S_{3,1,1,4} - 14336S_{3,1,4,1} + 8192S_{3,2,-3,1} \\
& +10240S_{3,2,-2,-2} + 8192S_{3,2,-2,2} - 6144S_{3,3,-2,1} - 18432S_{3,4,1,1} - 63488S_{4,-3,1,1} \\
& +8192S_{4,-2,-2,1} + 4096S_{4,-2,1,-2} - 38912S_{4,-2,1,2} - 38912S_{4,-2,2,1} - 65536S_{4,1,-3,1} \\
& +8192S_{4,1,-2,-2} - 24576S_{4,1,-2,2} - 73728S_{4,1,1,-3} - 18432S_{4,1,1,3} - 32768S_{4,1,2,-2} \\
& -40960S_{4,1,2,2} - 18432S_{4,1,3,1} - 40960S_{4,2,-2,1} - 32768S_{4,2,1,-2} - 40960S_{4,2,1,2} \\
& -40960S_{4,2,2,1} - 18432S_{4,3,1,1} - 73728S_{5,-2,1,1} - 98304S_{5,1,-2,1} - 65536S_{5,1,1,-2} \\
& -81920S_{5,1,1,2} - 81920S_{5,1,2,1} - 81920S_{5,2,1,1} - 45056S_{6,1,1,1} + 118784S_{-4,-2,1,1,1} \\
& +86016S_{-4,1,-2,1,1} + 24576S_{-4,1,1,-2,1} + 4096S_{-4,1,1,1,-2} + 311296S_{-3,-3,1,1,1} \\
& +180224S_{-3,-2,-2,1,1} + 180224S_{-3,-2,1,-2,1} + 131072S_{-3,-2,1,1,-2} + 163840S_{-3,-2,1,1,2} \\
& +163840S_{-3,-2,1,2,1} + 163840S_{-3,-2,2,1,1} + 245760S_{-3,1,-3,1,1} + 196608S_{-3,1,-2,-2,1} \\
& +122880S_{-3,1,-2,1,-2} + 147456S_{-3,1,-2,1,2} + 147456S_{-3,1,-2,2,1} + 122880S_{-3,1,1,-3,1} \\
& +81920S_{-3,1,1,-2,-2} + 98304S_{-3,1,1,-2,2} + 24576S_{-3,1,1,1,-3} + 24576S_{-3,1,2,-2,1} \\
& +98304S_{-3,2,-2,1,1} + 24576S_{-3,2,1,-2,1} + 118784S_{-2,-4,1,1,1} + 167936S_{-2,-3,-2,1,1} \\
& +172032S_{-2,-3,1,-2,1} + 131072S_{-2,-3,1,1,-2} + 163840S_{-2,-3,1,1,2} + 163840S_{-2,-3,1,2,1} \\
& +163840S_{-2,-3,2,1,1} + 159744S_{-2,-2,-3,1,1} + 24576S_{-2,-2,-2,-2,1} \\
& +24576S_{-2,-2,-2,1,-2} + 77824S_{-2,-2,-2,1,2} + 77824S_{-2,-2,-2,2,1} + 163840S_{-2,-2,1,-3,1} \\
& +24576S_{-2,-2,1,-2,-2} + 81920S_{-2,-2,1,-2,2} + 147456S_{-2,-2,1,1,-3} + 36864S_{-2,-2,1,1,3} \\
& +65536S_{-2,-2,1,2,-2} + 81920S_{-2,-2,1,2,2} + 36864S_{-2,-2,1,3,1} + 81920S_{-2,-2,2,-2,1} \\
& +65536S_{-2,-2,2,1,-2} + 81920S_{-2,-2,2,1,2} + 81920S_{-2,-2,2,2,1} + 36864S_{-2,-2,3,1,1} \\
& +86016S_{-2,1,-4,1,1} + 192512S_{-2,1,-3,-2,1} + 122880S_{-2,1,-3,1,-2} \\
& +147456S_{-2,1,-3,1,2} + 147456S_{-2,1,-3,2,1} + 176128S_{-2,1,-2,-3,1} + 24576S_{-2,1,-2,-2,-2} \\
& +86016S_{-2,1,-2,-2,2} + 155648S_{-2,1,-2,1,-3} + 36864S_{-2,1,-2,1,3} + 65536S_{-2,1,-2,2,-2}
\end{aligned}$$

$$\begin{aligned}
&+81920S_{-2,1,-2,2,2} + 36864S_{-2,1,-2,3,1} + 40960S_{-2,1,1,-4,1} + 86016S_{-2,1,1,-3,-2} \\
&+98304S_{-2,1,1,-3,2} + 159744S_{-2,1,1,-2,-3} + 24576S_{-2,1,1,-2,3} + 16384S_{-2,1,1,1,-4} \\
&+28672S_{-2,1,1,1,4} + 102400S_{-2,1,1,4,1} + 32768S_{-2,1,2,-3,1} + 28672S_{-2,1,2,-2,-2} \\
&+32768S_{-2,1,2,-2,2} - 8192S_{-2,1,3,-2,1} + 86016S_{-2,1,4,1,1} + 98304S_{-2,2,-3,1,1} \\
&+102400S_{-2,2,-2,-2,1} + 57344S_{-2,2,-2,1,-2} + 65536S_{-2,2,-2,1,2} + 65536S_{-2,2,-2,2,1} \\
&+32768S_{-2,2,1,-3,1} + 28672S_{-2,2,1,-2,-2} + 32768S_{-2,2,1,-2,2} + 4096S_{-2,3,-2,1,1} \\
&-8192S_{-2,3,1,-2,1} + 77824S_{-2,4,1,1,1} + 118784S_{1,-4,-2,1,1} + 49152S_{1,-4,1,-2,1} \\
&+8192S_{1,-4,1,1,-2} + 311296S_{1,-3,-3,1,1} + 192512S_{1,-3,-2,-2,1} + 139264S_{1,-3,-2,1,-2} \\
&+163840S_{1,-3,-2,1,2} + 163840S_{1,-3,-2,2,1} + 221184S_{1,-3,1,-3,1} + 118784S_{1,-3,1,-2,-2} \\
&+147456S_{1,-3,1,-2,2} + 81920S_{1,-3,1,1,-3} + 8192S_{1,-3,1,2,-2} + 73728S_{1,-3,2,-2,1} \\
&+8192S_{1,-3,2,1,-2} + 118784S_{1,-2,-4,1,1} + 184320S_{1,-2,-3,-2,1} + 131072S_{1,-2,-3,1,-2} \\
&+163840S_{1,-2,-3,1,2} + 163840S_{1,-2,-3,2,1} + 184320S_{1,-2,-2,-3,1} + 24576S_{1,-2,-2,-2,-2} \\
&+94208S_{1,-2,-2,-2,2} + 155648S_{1,-2,-2,1,-3} + 36864S_{1,-2,-2,1,3} + 65536S_{1,-2,-2,2,-2} \\
&+81920S_{1,-2,-2,2,2} + 36864S_{1,-2,-2,3,1} + 81920S_{1,-2,1,-4,1} + 118784S_{1,-2,1,-3,-2} \\
&+147456S_{1,-2,1,-3,2} + 159744S_{1,-2,1,-2,-3} + 36864S_{1,-2,1,-2,3} + 40960S_{1,-2,1,1,-4} \\
&+53248S_{1,-2,1,1,4} + 24576S_{1,-2,1,2,-3} - 4096S_{1,-2,1,3,-2} + 94208S_{1,-2,1,4,1} \\
&+90112S_{1,-2,2,-3,1} + 53248S_{1,-2,2,-2,-2} + 65536S_{1,-2,2,-2,2} + 24576S_{1,-2,2,1,-3} \\
&-4096S_{1,-2,3,1,-2} + 94208S_{1,-2,4,1,1} - 32768S_{1,1,-5,1,1} + 77824S_{1,1,-4,-2,1} \\
&+12288S_{1,1,-4,1,-2} + 8192S_{1,1,-4,1,2} + 8192S_{1,1,-4,2,1} + 278528S_{1,1,-3,-3,1} \\
&+139264S_{1,1,-3,-2,-2} + 163840S_{1,1,-3,-2,2} + 147456S_{1,1,-3,1,-3} - 8192S_{1,1,-3,1,3} \\
&+32768S_{1,1,-3,2,-2} - 8192S_{1,1,-3,3,1} + 110592S_{1,1,-2,-4,1} + 139264S_{1,1,-2,-3,-2} \\
&+163840S_{1,1,-2,-3,2} + 172032S_{1,1,-2,-2,-3} + 36864S_{1,1,-2,-2,3} + 61440S_{1,1,-2,1,-4} \\
&+65536S_{1,1,-2,1,4} + 57344S_{1,1,-2,2,-3} - 12288S_{1,1,-2,3,-2} + 98304S_{1,1,-2,4,1} \\
&-32768S_{1,1,1,-5,1} + 4096S_{1,1,1,-4,-2} - 8192S_{1,1,1,-4,2} + 221184S_{1,1,1,-3,-3} \\
&-8192S_{1,1,1,-3,3} + 69632S_{1,1,1,-2,-4} + 57344S_{1,1,1,-2,4} - 16384S_{1,1,1,1,-5} + 65536S_{1,1,1,1,5} \\
&+73728S_{1,1,1,4,-2} + 81920S_{1,1,1,4,2} + 131072S_{1,1,1,5,1} + 32768S_{1,1,2,-3,-2} \\
&+16384S_{1,1,2,-3,2} + 131072S_{1,1,2,-2,-3} + 98304S_{1,1,2,4,1} - 16384S_{1,1,3,-3,1} \\
&-20480S_{1,1,3,-2,-2} - 16384S_{1,1,3,-2,2} + 98304S_{1,1,4,-2,1} + 73728S_{1,1,4,1,-2} \\
&+81920S_{1,1,4,1,2} + 81920S_{1,1,4,2,1} + 163840S_{1,1,5,1,1} - 8192S_{1,2,-4,1,1} + 163840S_{1,2,-3,-2,1} \\
&+57344S_{1,2,-3,1,-2} + 16384S_{1,2,-3,1,2} + 16384S_{1,2,-3,2,1} + 147456S_{1,2,-2,-3,1} \\
&+73728S_{1,2,-2,-2,-2} + 81920S_{1,2,-2,-2,2} + 90112S_{1,2,-2,1,-3} - 8192S_{1,2,-2,1,3} \\
&+24576S_{1,2,-2,2,-2} - 8192S_{1,2,-2,3,1} + 32768S_{1,2,1,-3,-2} + 16384S_{1,2,1,-3,2} \\
&+131072S_{1,2,1,-2,-3} + 98304S_{1,2,1,4,1} + 81920S_{1,2,4,1,1} - 8192S_{1,3,-3,1,1} + 28672S_{1,3,-2,-2,1} \\
&+8192S_{1,3,-2,1,2} + 8192S_{1,3,-2,2,1} - 16384S_{1,3,1,-3,1} - 20480S_{1,3,1,-2,-2} - 16384S_{1,3,1,-2,2}
\end{aligned}$$

$$\begin{aligned}
& +61440S_{1,4,-2,1,1} + 90112S_{1,4,1,-2,1} + 65536S_{1,4,1,1,-2} + 81920S_{1,4,1,1,2} + 81920S_{1,4,1,2,1} \\
& +81920S_{1,4,2,1,1} + 163840S_{1,5,1,1,1} + 8192S_{2,-4,1,1,1} + 163840S_{2,-3,-2,1,1} \\
& +114688S_{2,-3,1,-2,1} + 32768S_{2,-3,1,1,-2} + 163840S_{2,-2,-3,1,1} + 98304S_{2,-2,-2,-2,1} \\
& +73728S_{2,-2,-2,1,-2} + 81920S_{2,-2,-2,1,2} + 81920S_{2,-2,-2,2,1} + 131072S_{2,-2,1,-3,1} \\
& +65536S_{2,-2,1,-2,-2} + 81920S_{2,-2,1,-2,2} + 57344S_{2,-2,1,1,-3} + 8192S_{2,-2,1,2,-2} \\
& +49152S_{2,-2,2,-2,1} + 8192S_{2,-2,2,1,-2} - 8192S_{2,1,-4,1,1} + 163840S_{2,1,-3,-2,1} \\
& +57344S_{2,1,-3,1,-2} + 16384S_{2,1,-3,1,2} + 16384S_{2,1,-3,2,1} + 147456S_{2,1,-2,-3,1} \\
& +73728S_{2,1,-2,-2,-2} + 81920S_{2,1,-2,-2,2} + 90112S_{2,1,-2,1,-3} - 8192S_{2,1,-2,1,3} \\
& +24576S_{2,1,-2,2,-2} - 8192S_{2,1,-2,3,1} + 32768S_{2,1,1,-3,-2} + 16384S_{2,1,1,-3,2} \\
& +131072S_{2,1,1,-2,-3} + 98304S_{2,1,1,4,1} + 81920S_{2,1,4,1,1} + 16384S_{2,2,-3,1,1} \\
& +98304S_{2,2,-2,-2,1} + 32768S_{2,2,-2,1,-2} + 16384S_{2,2,-2,1,2} + 16384S_{2,2,-2,2,1} \\
& -16384S_{2,3,-2,1,1} + 81920S_{2,4,1,1,1} + 8192S_{3,-3,1,1,1} + 36864S_{3,-2,-2,1,1} \\
& +16384S_{3,-2,1,-2,1} + 4096S_{3,-2,1,1,-2} - 8192S_{3,1,-3,1,1} + 28672S_{3,1,-2,-2,1} \\
& +8192S_{3,1,-2,1,2} + 8192S_{3,1,-2,2,1} - 16384S_{3,1,1,-3,1} - 20480S_{3,1,1,-2,-2} - 16384S_{3,1,1,-2,2} \\
& -16384S_{3,2,-2,1,1} + 77824S_{4,-2,1,1,1} + 49152S_{4,1,-2,1,1} + 81920S_{4,1,1,-2,1} + 65536S_{4,1,1,1,-2} \\
& +81920S_{4,1,1,1,2} + 81920S_{4,1,1,2,1} + 81920S_{4,1,2,1,1} + 81920S_{4,2,1,1,1} + 163840S_{5,1,1,1,1} \\
& -327680S_{-3,-2,1,1,1,1} - 294912S_{-3,1,-2,1,1,1} - 196608S_{-3,1,1,-2,1,1} - 49152S_{-3,1,1,1,-2,1} \\
& -327680S_{-2,-3,1,1,1,1} - 155648S_{-2,-2,-2,1,1,1} - 163840S_{-2,-2,1,-2,1,1} \\
& -163840S_{-2,-2,1,1,-2,1} - 131072S_{-2,-2,1,1,1,-2} - 163840S_{-2,-2,1,1,1,2} - 163840S_{-2,-2,1,1,2,1} \\
& -163840S_{-2,-2,1,2,1,1} - 163840S_{-2,-2,2,1,1,1} - 294912S_{-2,1,-3,1,1,1} - 172032S_{-2,1,-2,-2,1,1} \\
& -180224S_{-2,1,-2,1,-2,1} - 131072S_{-2,1,-2,1,1,-2} - 163840S_{-2,1,-2,1,1,2} - 163840S_{-2,1,-2,1,2,1} \\
& -163840S_{-2,1,-2,2,1,1} - 196608S_{-2,1,1,-3,1,1} - 204800S_{-2,1,1,-2,-2,1} - 114688S_{-2,1,1,-2,1,-2} \\
& -131072S_{-2,1,1,-2,1,2} - 131072S_{-2,1,1,-2,2,1} - 65536S_{-2,1,1,1,-3,1} - 57344S_{-2,1,1,1,-2,-2} \\
& -65536S_{-2,1,1,1,-2,2} + S_2((1024S_{-3} + 4096S_3)S_{-2}^2 + (11264S_{-5} + 5120S_5 - 8192S_{-4,1} \\
& -6144S_{-3,2} - 8192S_{-2,3} + 2048S_{4,1} + 12288S_{-3,1,1} - 4096S_{-2,-2,1} + 12288S_{-2,1,2} \\
& +12288S_{-2,2,1} - 24576S_{-2,1,1,1})S_{-2} + 8192S_{-7} + 9216S_7 - 16384S_{-6,1} - 6144S_{-5,-2} \\
& -16384S_{-5,2} - 1024S_{-4,-3} - 17408S_{-4,3} - 15360S_{-3,4} - 18432S_{-2,5} - 5120S_{4,3} \\
& +4096S_{5,2} + 6144S_{6,1} + 32768S_{-5,1,1} - 6144S_{-4,-2,1} + 36864S_{-4,1,2} + 36864S_{-4,2,1} \\
& -4096S_{-3,-3,1} - 2048S_{-3,-2,-2} - 4096S_{-3,-2,2} + 36864S_{-3,1,3} + 40960S_{-3,2,2} \\
& +36864S_{-3,3,1} + 2048S_{-2,-4,1} - 8192S_{-2,-3,2} + 10240S_{-2,-2,3} + S_{-2,1}(-4096S_{-4} \\
& -8192S_4 + 12288S_{-3,1} + 16384S_{-2,2} - 8192S_{-2,1,1}) + S_{-3}(6144S_{-4} + 3072S_4 \\
& -6144S_{-3,1} - 4096S_{-2,2} + 12288S_{-2,1,1}) + S_3(10240S_{-4} + 3072S_4 - \frac{47104S_{-3,1}}{3}
\end{aligned}$$

$$\begin{aligned}
& -\frac{40960S_{-2,2}}{3} + \frac{69632}{3}S_{-2,1,1}) + 34816S_{-2,1,4} + 36864S_{-2,2,3} + 36864S_{-2,3,2} \\
& + 32768S_{-2,4,1} - 4096S_{4,1,2} - 4096S_{4,2,1} - 73728S_{-4,1,1,1} - 81920S_{-3,1,1,2} \\
& - 81920S_{-3,1,2,1} - 81920S_{-3,2,1,1} + 24576S_{-2,-3,1,1} + 4096S_{-2,-2,-2,1} + 8192S_{-2,-2,1,2} \\
& + 8192S_{-2,-2,2,1} - 8192S_{-2,1,1,-3} - 73728S_{-2,1,1,3} - 81920S_{-2,1,2,2} - 73728S_{-2,1,3,1} \\
& - 8192S_{-2,2,-2,1} - 81920S_{-2,2,1,2} - 81920S_{-2,2,2,1} - 73728S_{-2,3,1,1} + 24576S_{4,1,1,1} \\
& + 163840S_{-3,1,1,1,1} - 49152S_{-2,-2,1,1,1} - 16384S_{-2,1,-2,1,1} + 163840S_{-2,1,1,1,2} \\
& + 163840S_{-2,1,1,2,1} + 163840S_{-2,1,2,1,1} + 163840S_{-2,2,1,1,1} - 327680S_{-2,1,1,1,1,1}) \\
& - 65536S_{-2,1,2,-2,1,1} - 131072S_{-2,2,-2,1,1,1} - 65536S_{-2,2,1,-2,1,1} - 327680S_{1,-3,-2,1,1,1} \\
& - 294912S_{1,-3,1,-2,1,1} - 147456S_{1,-3,1,1,-2,1} - 16384S_{1,-3,1,1,1,-2} - 327680S_{1,-2,-3,1,1,1} \\
& - 188416S_{1,-2,-2,-2,1,1} - 180224S_{1,-2,-2,1,-2,1} - 131072S_{1,-2,-2,1,1,-2} \\
& - 163840S_{1,-2,-2,1,1,2} - 163840S_{1,-2,-2,1,2,1} - 163840S_{1,-2,-2,2,1,1} - 294912S_{1,-2,1,-3,1,1} \\
& - 188416S_{1,-2,1,-2,-2,1} - 131072S_{1,-2,1,-2,1,-2} - 163840S_{1,-2,1,-2,1,2} - 163840S_{1,-2,1,-2,2,1} \\
& - 180224S_{1,-2,1,1,-3,1} - 106496S_{1,-2,1,1,-2,-2} - 131072S_{1,-2,1,1,-2,2} - 49152S_{1,-2,1,1,1,-3} \\
& - 49152S_{1,-2,1,2,-2,1} - 131072S_{1,-2,2,-2,1,1} - 49152S_{1,-2,2,1,-2,1} - 16384S_{1,1,-4,1,1,1} \\
& - 327680S_{1,1,-3,-2,1,1} - 229376S_{1,1,-3,1,-2,1} - 65536S_{1,1,-3,1,1,-2} - 327680S_{1,1,-2,-3,1,1} \\
& - 196608S_{1,1,-2,-2,-2,1} - 147456S_{1,1,-2,-2,1,-2} - 163840S_{1,1,-2,-2,1,2} - 163840S_{1,1,-2,-2,2,1} \\
& - 262144S_{1,1,-2,1,-3,1} - 131072S_{1,1,-2,1,-2,-2} - 163840S_{1,1,-2,1,-2,2} - 114688S_{1,1,-2,1,1,-3} \\
& - 16384S_{1,1,-2,1,2,-2} - 98304S_{1,1,-2,2,-2,1} - 16384S_{1,1,-2,2,1,-2} + 16384S_{1,1,1,-4,1,1} \\
& - 327680S_{1,1,1,-3,-2,1} - 114688S_{1,1,1,-3,1,-2} - 32768S_{1,1,1,-3,1,2} - 32768S_{1,1,1,-3,2,1} \\
& - 294912S_{1,1,1,-2,-3,1} - 147456S_{1,1,1,-2,-2,-2} - 163840S_{1,1,1,-2,-2,2} - 180224S_{1,1,1,-2,1,-3} \\
& + 16384S_{1,1,1,-2,1,3} - 49152S_{1,1,1,-2,2,-2} + 16384S_{1,1,1,-2,3,1} - 65536S_{1,1,1,1,-3,-2} \\
& - 32768S_{1,1,1,1,-3,2} - 262144S_{1,1,1,1,-2,-3} - 196608S_{1,1,1,1,4,1} - 163840S_{1,1,1,4,1,1} \\
& - 32768S_{1,1,2,-3,1,1} - 196608S_{1,1,2,-2,-2,1} - 65536S_{1,1,2,-2,1,-2} - 32768S_{1,1,2,-2,1,2} \\
& - 32768S_{1,1,2,-2,2,1} + 32768S_{1,1,3,-2,1,1} - 163840S_{1,1,4,1,1,1} - 32768S_{1,2,-3,1,1,1} \\
& - 163840S_{1,2,-2,-2,1,1} - 131072S_{1,2,-2,1,-2,1} - 49152S_{1,2,-2,1,1,-2} - 32768S_{1,2,1,-3,1,1} \\
& - 196608S_{1,2,1,-2,-2,1} - 65536S_{1,2,1,-2,1,-2} - 32768S_{1,2,1,-2,1,2} - 32768S_{1,2,1,-2,2,1} \\
& - 16384S_{1,3,-2,1,1,1} + 32768S_{1,3,1,-2,1,1} - 163840S_{1,4,1,1,1,1} - 163840S_{2,-2,-2,1,1,1} \\
& - 163840S_{2,-2,1,-2,1,1} - 98304S_{2,-2,1,1,-2,1} - 16384S_{2,-2,1,1,1,-2} - 32768S_{2,1,-3,1,1,1} \\
& - 163840S_{2,1,-2,-2,1,1} - 131072S_{2,1,-2,1,-2,1} - 49152S_{2,1,-2,1,1,-2} - 32768S_{2,1,1,-3,1,1} \\
& - 196608S_{2,1,1,-2,-2,1} - 65536S_{2,1,1,-2,1,-2} - 32768S_{2,1,1,-2,1,2} - 32768S_{2,1,1,-2,2,1} \\
& - 32768S_{2,2,-2,1,1,1} - 16384S_{3,1,-2,1,1,1} + 32768S_{3,1,1,-2,1,1} - 163840S_{4,1,1,1,1,1} \\
& + 327680S_{-2,-2,1,1,1,1,1} + 327680S_{-2,1,-2,1,1,1,1} + 262144S_{-2,1,1,-2,1,1,1}
\end{aligned}$$

$$\begin{aligned}
& +131072S_{-2,1,1,1,-2,1,1} + 327680S_{1,-2,-2,1,1,1,1} + 327680S_{1,-2,1,-2,1,1,1} \\
& +262144S_{1,-2,1,1,-2,1,1} + 98304S_{1,-2,1,1,1,-2,1} + 327680S_{1,1,-2,-2,1,1,1} \\
& +327680S_{1,1,-2,1,-2,1,1} + 196608S_{1,1,-2,1,1,-2,1} + 32768S_{1,1,-2,1,1,1,-2} + 65536S_{1,1,1,-3,1,1,1} \\
& +327680S_{1,1,1,-2,-2,1,1} + 262144S_{1,1,1,-2,1,-2,1} + 98304S_{1,1,1,-2,1,1,-2} + 65536S_{1,1,1,1,-3,1,1} \\
& +393216S_{1,1,1,1,-2,-2,1} + 131072S_{1,1,1,1,-2,1,-2} + 65536S_{1,1,1,1,-2,1,2} + 65536S_{1,1,1,1,-2,2,1} \\
& +65536S_{1,1,2,-2,1,1,1} + 65536S_{1,2,1,-2,1,1,1} + 65536S_{2,1,1,-2,1,1,1} - 131072S_{1,1,1,1,-2,1,1,1} \\
& +512 \left(4S_{-2,1}S_{-3} - S_{-3}^2 + S_3^2 - 4S_{-2,1}^2 + S_1^2 \left(2S_{-2}^2 - 4S_{-4} + 6S_4 + 16S_{-3,1} + 12S_{-2,2} \right. \right. \\
& \left. \left. - 16S_{-2,1,1} \right) + S_1 \left(-2S_{-5} - 4S_{-3}S_2 + 4S_{-2}S_3 + 4S_2S_3 + 6S_5 + 8S_{-4,1} - 4S_{-3,-2} \right. \right. \\
& \left. \left. + 12S_{-3,2} + 8S_{-2}S_{-2,1} + 8S_2S_{-2,1} + 8S_{-2,3} + 4S_{4,1} - 24S_{-3,1,1} - 8S_{-2,-2,1} - 24S_{-2,1,2} \right. \right. \\
& \left. \left. - 24S_{-2,2,1} + 48S_{-2,1,1,1} \right) \right) \zeta(3) \\
& +2560 S_1(S_3 - S_{-3} + 2 S_{-2,1}) \zeta(5)
\end{aligned}$$

Table 2: The ABA contribution to the five-loop anomalous dimension of twist-two operators γ_{10}^{ABA} .

B Eigenvalues of $\hat{S}_{\text{su}(2|2),Q}$

B.1 Bosonic eigenvalues

The bosonic eigenvalues correspond to the bosonic part of the $\hat{S}_{\text{su}(2|2),Q}$ matrix. It turns out that their structure becomes concise and transparent upon parametrizing B_1 with

$$j^+ = j - \frac{1}{2}(-2 + Q + i q), \quad (\text{B.1})$$

and B_2 with

$$j^- = j - \frac{1}{2}(Q + i q). \quad (\text{B.2})$$

Below we present their one-, two- and three-loop expansion coefficients using j^+ and j^- instead of j .

B.1.1 One-loop

At the one-loop order one finds

$$B_{1,0} = B_{1,0}^0 + B_{1,0}^1 j^+, \quad B_{2,0} = B_{2,0}^0 + B_{2,0}^1 j^-,$$

with the corresponding coefficients

$$B_{1,0}^0 = \frac{i-2u}{q-iQ-2u+i}, \quad B_{1,0}^1 = \frac{2(2iu+1)}{(q-iQ-2u+i)(2u+i)},$$

$$B_{2,0}^0 = \frac{i-2u}{q-iQ-2u+i}, \quad B_{2,0}^1 = \frac{2i}{q-iQ-2u+i}.$$

B.1.2 Two-loop

At the next order inverse powers of j^+ and j^- appear

$$B_{1,2} = \frac{B_{1,2}^{-1}}{j^+} + B_{1,2}^0 + B_{1,2}^1 j^+, \quad B_{2,2} = \frac{B_{2,2}^{-1}}{j^-} + B_{2,2}^0 + B_{2,2}^1 j^-. \quad (\text{B.3})$$

The corresponding coefficients are given by

$$B_{1,2}^{-1} = \frac{4}{(q-iQ-2u+i)(2u+i)}, \quad B_{1,2}^0 = \frac{8i(q^2 - i(i-2u)^2 q + Q^2 + (i-2u)^2)}{(q+iQ)(q-iQ-2u+i)^2(4u^2+1)},$$

$$B_{1,2}^1 = \frac{16}{(q^2+Q^2)(q-iQ-2u+i)^2(2u-i)(2u+i)^3} \left(4uq^3 + \right.$$

$$\left. (-4u^2 - 4i(Q-1)u+1)q^2 + 2(2Q^2 + (i-2u)^2(1-2iu))uq \right.$$

$$\left. + Q(Q-2iu-1)(4u(u-iQ)+1) \right),$$

$$B_{2,2}^{-1} = -\frac{4}{(q-iQ-2u+i)(2u+i)}, \quad B_{2,2}^1 = \frac{16(q^2 + 2(-2iu-1)uq + Q(Q-2iu-1))}{(q^2+Q^2)(q-iQ-2u+i)^2(4u^2+1)},$$

$$B_{2,2}^0 = \frac{1}{(q^2+Q^2)(q-iQ-2u+i)^2(4u^2+1)} \left(8i(q^3 - i(4u^2+Q+1)q^2 \right.$$

$$\left. + (Q^2 - Q - 4(Q+1)u^2 + 4iu+1)q - iQ(Q^2 + (i-2u)^2) \right).$$

B.1.3 Three-loop

At the three-loop order one derives

$$B_{1,4} = \frac{B_{1,4}^{-3}}{(j^+)^3} + \frac{B_{1,4}^{-2}}{(j^+)^2} + \frac{B_{1,4}^{-1}}{j^+} + B_{1,4}^0 + B_{1,4}^1 j^+, \quad (\text{B.4})$$

$$B_{2,4} = \frac{B_{2,4}^{-3}}{(j^-)^3} + \frac{B_{2,4}^{-2}}{(j^-)^2} + \frac{B_{2,4}^{-1}}{j^-} + B_{2,4}^0 + B_{2,4}^1 j^-, \quad (\text{B.5})$$

with the corresponding coefficients

$$B_{1,4}^{-3} = -\frac{4}{(q - iQ - 2u + i)(2u + i)}, \quad B_{1,4}^{-2} = \frac{16iq}{(q^2 + Q^2)(q - iQ - 2u + i)(2u + i)},$$

$$B_{1,4}^{-1} = \frac{32}{(q^2 + Q^2)(q - iQ - 2u + i)^2(2u - i)(2u + i)^3} \left((-8qu^3 - 4(q^2 + Q^2 + Q)u^2 + 2(q^3 - iQq^2 + (Q^2 - 1)q - iQ^3)u + q^2 + (Q - 1)Q) \right),$$

$$B_{1,4}^0 = \frac{32i}{(q - iQ)(q + iQ)^2(q - iQ - 2u + i)^3(4u^2 + 1)^3} \left((12u^2 - 1)q^5 + (-56u^3 - 12i(Q - 3)u^2 + 10u + i(Q - 3))q^4 + 2(16u^4 + 16i(Q - 3)u^3 + 4(Q(3Q + 4) - 10)u^2 - 4i(Q - 3)u - Q^2 + 1)q^3 - i((24u^2 - 2)Q^3 - 2i(2u - i)(20u^2 - 3)Q^2 + 2(i - 2u)^2(4u(u - i) + 1)Q - (2u - i)^3(6u(4u^2 + 2iu + 1) - i))q^2 + ((12u^2 - 1)Q^4 + 8i(i - 2u)^2uQ^3 + 2(i - 2u)^2(4u(u - 2i) - 1)Q^2 + (i - 2u)^4(4u(u + 2i) - 3)Q + 2(i - 2u)^4(1 - iu(4u(u - i) + 5)))q - i(Q - 2iu - 1)^2(Q + 2iu + 1) \times (-16u^4 + 4(Q(3Q + 2) - 2)u^2 - (Q - 1)^2) \right),$$

$$B_{1,4}^1 = \frac{64}{(q^2 + Q^2)^2(q - iQ - 2u + i)^3(2u - i)^3(2u + i)^5} \left(4u(4u(3u - i) - 3)q^6 - (8u(Q(4(3iu + 1)u - 3i) + u(2u(9u - 8i) - 11) + 4i) + 1)q^5 + (-96u^5 + 48i(3Q - 1)u^4 + 16(Q - 1)(3Q + 11)u^3 - 8i(Q - 1)(2Q + 13)u^2 - 4Q(3Q + 8)u + 26u + iQ - 3i)q^4 + 2(8u(4(-3iu - 1)u + 3i)Q^3 + (8u(u(2(8i - 9u)u + 11) - 4i) - 1)Q^2 + 4iu(4u^2 + 1)^2Q + 2u(2u - i)^3(4u(u - i) + 3))q^3 + 2(2u(4(i - 3u)u + 3)Q^4 + (2iu + 1)(2u(2u(18u - 7i) - 15) + i)Q^3 - (2u - i)^3(4u(u + 3i) + 3)Q^2 + 2(-2iu - 1)^3(4u^3 - u - i)Q + i(i - 2u)^4u(2u + i)^2(6u + i))q^2 + (8u(4(-3iu - 1)u + 3i)Q^5 + (8u(u(2(8i - 9u)u + 11) - 4i) - 1)Q^4 + 8iu(4u^2 + 1)^2Q^3 + 4u(2u - i)^3(4u(u - i) + 3)Q^2 + (i - 2u)^4(2u + i)^2(4u(u + 3i) - 1)Q + 2(2iu - 1)^3u(2u - i)^5)q - Q(Q - 2iu - 1)^2(4u(4u(3u - i) - 3)Q^3 + i(i - 2u)^2(4u(3u + i) + 1)Q^2 - (2u - i)^3(2u + i)^2Q - i(4u^2 + 1)^3) \right),$$

$$B_{2,4}^{-3} = \frac{4}{(q - iQ - 2u + i)(2u + i)}, \quad B_{2,4}^{-2} = -\frac{16iq}{(q^2 + Q^2)(q - iQ - 2u + i)(2u + i)},$$

$$B_{2,4}^{-1} = \frac{32}{(q^2 + Q^2)(q - iQ - 2u + i)^2(2u - i)(2u + i)^3} \left((8qu^3 + 4(q^2 + Q^2 + Q)u^2 + 2(-q^3 + iQq^2 - Q^2q + q + iQ^3)u - q^2 - Q^2 + Q) \right),$$

$$B_{2,4}^0 = \frac{32i}{(q^2 + Q^2)^2(q - iQ - 2u + i)^3(4u^2 + 1)^3} \left((12u^2 - 1)q^6 + (-24u^3 - 4i(6Q - 1)u^2 + 2u + i(2Q - 3))q^5 + (-96u^4 + 8i(3Q + 8)u^3 + 4Q(3Q + 1)u^2 - 2iQu - Q(Q + 3) - 2)q^4 - i((48u^2 - 4)Q^3 + (4u(2(-6iu - 1)u + i) + 6)Q^2 - 4(4u^2 + 1)^2Q - (2u - i)^3(2u(2u(6u + 5i) - 5) + i))q^3 + ((1 - 12u^2)Q^4 + (4u(2(6iu + 1)u - i) - 6)Q^3 - 32(i - 2u)^2u^2Q^2 + 2(2u - i)^3(4u(u(4u + 7i) - 3) - 3i)Q + 2(i - 2u)^4(u(4(3 - iu)u + 7i) - 1))q^2 - i((24u^2 - 2)Q^5 + (2u(2(-6iu - 1)u + i) + 3)Q^4 - 4(4u^2 + 1)^2Q^3 + (2u - i)^3(2u(2u(2u + 7i) - 7) - 5i)Q^2 + (i - 2u)^4(1 - 2iu)(4u^2 - 3)Q - i(2u - i)^5(2u + i)^2)q - Q(Q - 2iu - 1)^2(Q + 2iu + 1)(-16u^4 + 4(Q(3Q + 2) - 2)u^2 - (Q - 1)^2) \right)$$

$$B_{2,4}^1 = \frac{64}{(q^2 + Q^2)^2(q - iQ - 2u + i)^3(4u^2 + 1)^3} \left((12u^2 - 1)q^5 + (-40u^3 - 4i(3Q - 5)u^2 + 6u + i(Q - 3))q^4 + 2(Q^2(12u^2 - 1) - 2(i - 2u)^2u(2u + 3i))q^3 - 2i((12u^2 - 1)Q^3 - i(2u - i)(20u^2 - 3)Q^2 - 2(i - 2u)^2(u(2u + i) - 1)Q - u(2u - i)^3(4u(3u + 2i) - 1))q^2 + ((12u^2 - 1)Q^4 - 4(i - 2u)^2u(2u + 3i)Q^2 + (2u - i)^3(2u + i) \times (4u(u + 3i) - 1)Q - 2i(i - 2u)^4u(2u + i)^2)q - iQ(Q - 2iu - 1)^2(-16u^4 + 8iQu^3 + 4(3Q^2 + Q - 2)u^2 + 2iQu - Q^2 + Q - 1) \right).$$

B.2 Fermionic eigenvalues

Fermionic eigenvalues are structurally simpler than the bosonic ones. In particular, they are linear functions of j .

B.2.1 One-loop

At the one-loop order one finds

$$F_0 = F_0^0 + F_0^1 j, \quad (\text{B.6})$$

with

$$F_0^0 = 1, \quad F_0^1 = -\frac{2}{iq + Q - 2iu - 1}.$$

B.2.2 Two-loop

The two-loop contribution to F is given by

$$F_2 = F_2^0 + F_2^1 j, \quad (\text{B.7})$$

with the coefficients

$$F_2^0 = -\frac{8i}{(q - iQ)(4u^2 + 1)}, \quad F_2^1 = \frac{16(q^2 + 2(-2iu - 1)uq + Q(Q - 2iu - 1))}{(q^2 + Q^2)(q - iQ - 2u + i)^2(4u^2 + 1)}.$$

B.2.3 Three-loop

At the three-loop order one determines

$$F_4 = F_4^0 + F_4^2 j. \quad (\text{B.8})$$

The corresponding coefficients are given by

$$F_4^0 = -\frac{32i}{(q - iQ)^2(q + iQ)(4u^2 + 1)^3} \left(-16u^4 + 8(q + iQ)u^3 \right. \\ \left. + 4(3Q^2 + Q + q(3q - i) - 2)u^2 + 2(q + iQ)u - Q^2 - q(q + i) + Q - 1 \right),$$

$$F_4^2 = \frac{64}{(q^2 + Q^2)^2(q - iQ - 2u + i)^3(4u^2 + 1)^3} \left((12u^2 - 1)q^5 \right. \\ + (-40u^3 - 4i(3Q - 5)u^2 + 6u + i(Q - 3))q^4 + 2(Q^2(12u^2 - 1) \\ - 2(i - 2u)^2u(2u + 3i))q^3 - 2i((12u^2 - 1)Q^3 - i(2u - i)(20u^2 - 3)Q^2 \\ - 2(i - 2u)^2(u(2u + i) - 1)Q - u(2u - i)^3(4u(3u + 2i) - 1))q^2 \\ + ((12u^2 - 1)Q^4 - 4(i - 2u)^2u(2u + 3i)Q^2 + (2u - i)^3(2u + i) \times \\ (4u(u + 3i) - 1)Q - 2i(i - 2u)^4u(2u + i)^2)q - iQ(Q - 2iu - 1)^2 \times \\ \left. (-16u^4 + 8iQu^3 + 4(3Q^2 + Q - 2)u^2 + 2iQu - Q^2 + Q - 1) \right).$$

C Diagonal elements of $G\partial\hat{S}_{\mathfrak{su}(2|2),Q}G^{-1}$

In this section we use similar notation to that in Appendix B.

C.1 Bosons

C.1.1 One-loop

At the one-loop order one finds

$$DB_{1,0} = DB_{1,0}^0 + DB_{1,0}^1 j^+, \quad DB_{2,0} = DB_{2,0}^0 + DB_{2,0}^1 j^-, \quad (C.1)$$

with the corresponding coefficients

$$DB_{1,0}^0 = \frac{(q - i(Q - 2))(2u - i)}{(q - iQ - 2u + i)^2(2u + i)}, \quad DB_{1,0}^1 = -\frac{4iu + 2}{(q - iQ - 2u + i)^2(2u + i)},$$

$$DB_{2,0}^0 = \frac{q - iQ}{(q - iQ - 2u + i)^2}, \quad DB_{2,0}^1 = -\frac{2i}{(q - iQ - 2u + i)^2}.$$

C.1.2 Two-loop

At the next order the structure becomes more involved

$$DB_{1,2} = \frac{DB_{1,2}^{-2}}{(j^+)^2} + \frac{DB_{1,2}^{-1}}{j^+} + DB_{1,2}^0 + DB_{1,2}^1 j^+, \quad (C.2)$$

$$DB_{2,2} = \frac{DB_{2,2}^{-2}}{(j^-)^2} + \frac{DB_{2,2}^{-1}}{j^-} + DB_{2,2}^0 + DB_{2,2}^1 j^-. \quad (C.3)$$

The expansion coefficients read

$$DB_{1,2}^{-2} = \frac{2i}{(q - iQ - 2u + i)(2u + i)}, \quad DB_{1,2}^{-1} = -\frac{4}{(q - iQ - 2u + i)^2(2u + i)},$$

$$DB_{1,2}^0 = -\frac{8i}{(q - iQ)(q + iQ)^2(q - iQ - 2u + i)^3(2u - i)(2u + i)^3} \left(4uq^5 \right. \\ \left. - 4i(Q - 2iu - 3)uq^4 + 2u(4Q^2 + 2u(2(1 - 6iu)u - 7i) - 3)q^3 \right. \\ \left. + (32iu^5 - 16(Q - 5)u^4 + 8iQu^3 - 4(4Q^2 + Q - 8)u^2 \right. \\ \left. - 2i(Q(4(Q - 3)Q + 3) + 1)u - 4Q + 3)q^2 + (4uQ^4 \right. \\ \left. + 2(u(-8iu^3 + 4u^2 - 6iu - 3) + i)Q^2 + 4(i - 2u)^2(2u + i)Q \right. \\ \left. - (2u - i)^3(2u + i)^2)q + Q(Q - 2iu - 1)(16(Q - 1)u^4 + 8iQu^3 \right. \\ \left. - 4(Q + 2)u^2 - 2iQ(2(Q - 2)Q - 1)u - 2Q - 1) \right),$$

$$DB_{1,2}^1 = -\frac{32}{(q^2 + Q^2)^2(q - iQ - 2u + i)^3(2u - i)(2u + i)^3} (2uq^5 \\ + (1 - 2i(Q - 1)u)q^4 + (4Q^2 + 3(i - 2u)^2(1 - 2iu))uq^3 \\ + (Q - 2iu - 1)(-4iuQ^2 + (8u^2 + 2)Q - (i - 2u)^2u(2u + i))q^2 \\ + Q(-8i(Q - 2)u^4 + 4(4 - 3Q)u^3 + 2iQu^2 + (2Q^3 - 3Q + 4)u \\ + i(Q - 1))q + Q^2(Q - 2iu - 1)(-2iuQ^2 + 4u^2Q \\ + Q + (i - 2u)^2u(2u + i))),$$

$$DB_{2,2}^{-2} = -\frac{2i}{(q - iQ - 2u + i)(2u + i)}, \quad DB_{2,2}^{-1} = \frac{4}{(q - iQ - 2u + i)^2(2u + i)},$$

$$DB_{2,2}^0 = -\frac{8i}{(q^2 + Q^2)^2(q - iQ - 2u + i)^3(4u^2 + 1)}(2q^5 - 2i(Q + u(6u - i) + 1)q^4 + (8iu^3 - 4(4Q + 1)u^2 + 10iu + 4Q^2 - 4Q + 3)q^3 + (-4iQ^3 + (2i - 4u)Q^2 + (2u - i)^3Q + (2u - i)^3)q^2 + Q(2Q^3 - i(2u - i)^3Q + 2(-2iu - 1)^3)q - iQ^2(Q - 2iu - 1)(2Q^2 + 2Q + 4(Q + 1)u^2 + 2i(Q - 2)u - 1)),$$

$$DB_{2,2}^1 = -\frac{32}{(q^2 + Q^2)^2(q - iQ - 2u + i)^3(4u^2 + 1)}\left(q^4 + 3(-2iu - 1)uq^3 + (Q - 2iu - 1)(2Q + (i - 2u)u)q^2 + Q(2u - i) \times (-Q - i(Q - 2)u + 1)q + Q^2(Q - 2iu - 1)(Q + u(2u - i))\right).$$

C.2 Fermions

The diagonal elements corresponding to the fermionic subspace are again simpler.

C.2.1 One-loop

At the one-loop order one finds

$$DF_0 = DF_0^1 j, \quad (C.4)$$

$$DF_0^1 = -\frac{2i}{(q - iQ - 2u + i)^2}.$$

C.3 Two-loop

At the next order one derives

$$DF_2 = DF_2^0 + DF_2^1 j, \quad (C.5)$$

with the following coefficients

$$DF_2^0 = \frac{8i}{(q - iQ)^2(4u^2 + 1)},$$

$$DF_2^1 = \frac{32}{(q^2 + Q^2)^2(-q + i(Q - 1) + 2u)^3(4u^2 + 1)^2}(uq^5 + (2u^2 - i(Q - 1)u + 1)q^4 + (2Q^2 + 3(i - 2u)^2(1 - 2iu))uq^3 + (Q - 2iu - 1)(-2iuQ^2 + (8u^2 + 2)Q - (i - 2u)^2u(2u + i))q^2 + Q(-8i(Q - 2)u^4 + 4(4 - 3Q)u^3 + 2iQu^2 + (Q^3 - 3Q + 4)u + i(Q - 1))q + Q^2(Q - 2iu - 1)(-iuQ^2 + 4u^2Q + Q + (i - 2u)^2u(2u + i))).$$

D Binomial sums

We define the binomial sums $\mathbb{S}_{i_1, \dots, i_k}$ through (see [40])

$$\mathbb{S}_{i_1, \dots, i_k}(N) = (-1)^N \sum_{j=1}^N (-1)^j \binom{N}{j} \binom{N+j}{j} S_{i_1, \dots, i_k}(j), \quad (\text{D.1})$$

where S_{i_1, \dots, i_k} is the nested harmonic sum defined in (2.4). The advantage of this basis is that only *positive* values of the indices i_1, \dots, i_k need to be considered. Moreover, these binomial sums appear in the real diagram calculations of the anomalous dimensions and coefficient functions in QCD [65, 66] and in the solution of the asymptotic all-loop Baxter equation for twist-two operators [42]. At the five-loop order there are 256 such sums, which is also the number of reciprocity-respecting harmonic sums defined in [46]⁸. We found the basis of binomial sums much easier to handle and implement, though a rigorous proof of the equivalence of the two basis of sums is unknown to us.

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⁸Relations between the binomial and the nested harmonic sums together with relations between the binomial and the reciprocity-respecting harmonic sums can be found under <http://thd.pnpi.spb.ru/~velizh/5loop/>

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